

GAUGE DEFORMATIONS FOR HOPF ALGEBRAS WITH THE DUAL CHEVALLEY PROPERTY

ALESSANDRO ARDIZZONI, MARGARET BEATTIE, AND CLAUDIA MENINI

ABSTRACT. Let A be a Hopf algebra over a field K of characteristic zero such that its coradical H is a finite dimensional sub-Hopf algebra. Our main theorem shows that there is a gauge transformation ζ on A such that $A^\zeta \cong Q \# H$ where A^ζ is the dual quasi-bialgebra obtained from A by twisting its multiplication by ζ , Q is a connected dual quasi-bialgebra in ${}^H_H\mathcal{YD}$ and $Q \# H$ is a dual quasi-bialgebra called the bosonization of Q by H .

CONTENTS

1. Introduction	1
2. Preliminaries	3
2.1. The category ${}^H_H\mathcal{YD}$	4
2.2. Some preliminary results	5
2.3. Pre-bialgebras with cocycle	7
2.4. Splitting data	8
2.5. Monoids of H -bilinear multibalanced and H -linear maps	9
3. Cohomology of (pre-)bialgebras	9
3.1. Cohomology of a K -bialgebra	9
3.2. Cohomology of a pre-bialgebra with cocycle	10
4. A new interpretation for the cocycle ξ for a pre-bialgebra R	13
5. The dual quasi-bialgebra A^{v_A}	18
5.1. Dual quasi-bialgebras and bosonization.	18
5.2. The dual quasi-bialgebra from a pre-bialgebra with cocycle	21
5.3. The bosonization of R^v with H .	23
6. The main theorem	24
References	24

1. INTRODUCTION

Let A be a Hopf algebra over a field K with characteristic 0 and suppose that A has the dual Chevalley property, in other words, $H := A_0$, the coradical of A , is a sub-Hopf algebra of A . For example pointed Hopf algebras have the dual Chevalley property. Moreover we assume that H is finite dimensional. Then if $\lambda \in H^*$ is the total integral for H , λ is left H -linear where the left H -action of H on itself is the adjoint action.

Under these conditions there is an H -bilinear coalgebra projection π from A to H which splits the inclusion so that $A \cong R \#_\xi H$ where (R, ξ) is a pre-bialgebra with cocycle in the category ${}^H_H\mathcal{YD}$ in the sense of [AMStu]. If $\xi = u_H \varepsilon_{R \otimes R}$, then R is a bialgebra in ${}^H_H\mathcal{YD}$ and A is isomorphic to a Radford biproduct. In [ABM], the question of finding a cocycle from $A \otimes A$ to K twisting a Hopf algebra A of the form $R \#_\xi H$ to a Radford biproduct was studied. A bijection between the

1991 *Mathematics Subject Classification.* Primary 16W30; Secondary 16S40.

Key words and phrases. Hopf algebra; dual Chevalley property; cocycle twist; dual quasi-Hopf algebra.

M. Beattie's research was supported by an NSERC Discovery Grant. Thanks to the University of Ferrara for the warm hospitality extended to M. Beattie during her visit of 2010. This paper was written while A. Ardizzoni and C. Menini were members of G.N.S.A.G.A. with partial financial support from M.I.U.R. (PRIN 2007).

sets of cocycles on A and on R shows that this question is equivalent to that of finding a cocycle $v : R \otimes R \rightarrow K$ twisting (R, ξ) to $(R^v, \xi_v = \varepsilon_{R \otimes R})$. The map $v = (\lambda \xi)^{-1}$ does give $\xi_v = \varepsilon_{R \otimes R}$ but has not been shown to be a cocycle; see [ABM, Section 5.2].

This paper shows that the correct setting for this problem is that of dual quasi-bialgebras. Twisting A by $v_A : A \otimes A \rightarrow K$, where v_A is the map $v := (\lambda \xi)^{-1}$ extended to $A \otimes A$, yields a dual quasi-bialgebra which is the biproduct of a dual quasi-bialgebra in ${}^H_H\mathcal{YD}$ and H .

The following is the main theorem of this paper:

Theorem I. Let A be a Hopf algebra over a field of characteristic zero such that the coradical of A is a sub-Hopf algebra. Assume the coradical H of A is finite dimensional so that H is semisimple. Then there is a gauge transformation ζ on A such that $A^\zeta \cong Q \# H$ where

- A^ζ is the dual quasi-bialgebra obtained from A by twisting its multiplication by ζ ;
- Q is a connected dual quasi-bialgebra in ${}^H_H\mathcal{YD}$;
- $Q \# H$ is a dual quasi-bialgebra called the bosonization of Q by H .

For H a cosemisimple Hopf algebra over a field K , let ${}^H_H\mathcal{YD}$ denote the braided monoidal category of Yetter-Drinfeld modules over H . Throughout this paper, all maps will be K -linear. Essentially the computations in this paper will construct functors between different categories and prove that the following diagram commutes.

$$(1) \quad \begin{array}{ccc} \mathcal{R} & \xrightarrow{B_1} & \mathcal{A} \\ T_1 \downarrow & & \downarrow T_2 \\ \mathcal{Q} & \xrightarrow{B_2} & \mathcal{B} \end{array}$$

where the categories $\mathcal{A}, \mathcal{B}, \mathcal{Q}, \mathcal{R}$ are defined as follows.

- \mathcal{R} has objects the pairs (R, ξ) where R is a connected pre-bialgebra with cocycle $\xi : R \otimes R \rightarrow H$ in ${}^H_H\mathcal{YD}$ as defined in Subsection 2.3. A morphism $f : (R, \xi) \rightarrow (R', \xi')$ in \mathcal{R} is a morphism of pre-bialgebras with cocycle in ${}^H_H\mathcal{YD}$, i.e., a coalgebra homomorphism $f : R \rightarrow R'$ in ${}^H_H\mathcal{YD}$ which is multiplicative and unitary and such that $\xi' \circ (f \otimes f) = \xi$ (see [AM, Definition 1.10]).
- \mathcal{A} has objects the pairs (A, γ) where A is a bialgebra with coradical H and $\gamma : A \otimes A \rightarrow K$ is a gauge transformation as defined in Subsection 5.1. A morphism $f : (A, \gamma) \rightarrow (A', \gamma')$ in \mathcal{A} is a morphism of bialgebras $f : A \rightarrow A'$ such that $f|_H = H$ and $\gamma'(f \otimes f) = \gamma$.
- \mathcal{Q} has objects the pairs (Q, α) where Q is a connected dual quasi-bialgebra in ${}^H_H\mathcal{YD}$ and $\alpha \in {}^H_H\mathcal{YD}(Q^{\otimes 3}, K)$ is the corresponding reassociator, see Definition 5.2. Morphisms in \mathcal{Q} are morphisms of dual quasi-bialgebras in ${}^H_H\mathcal{YD}$.
- \mathcal{B} has objects the pairs (B, β) where B is a dual quasi-bialgebra with coradical H and $\beta : B^{\otimes 3} \rightarrow K$ is the reassociator. A morphism $f : (B, \beta) \rightarrow (B', \beta')$ in \mathcal{B} is a morphism of dual quasi-bialgebras $f : B \rightarrow B'$ such that $f|_H = H$, see Subsection 5.1.

Assume that H has an ad-invariant integral λ . The functors T_i and B_i are twisting and bosonization functors where

- $B_1(R, \xi) := (R \#_\xi H, v_{R \#_\xi H})$ as defined in Subsection 5.3.
- $B_2(Q, \alpha) := (Q \# H, \alpha_{Q \# H})$ as defined in Proposition 5.3.
- $T_1(R, \xi) := (R^v, \partial_R^2 v)$ as defined in Proposition 5.5.
- $T_2(A, \gamma) := (A^\gamma, \partial_A^2 \gamma)$ as defined in Subsection 5.1.

Furthermore, for the bosonization functors B_i , we have $B_i(f) = f \otimes H$ for every morphism f in \mathcal{R}, \mathcal{Q} , and for the twisting functors T_i , we have $T_i(f) = f$ for every morphism f in \mathcal{R}, \mathcal{A} .

Proposition 5.7 says that $B_2 \circ T_1 = T_2 \circ B_1$. This result is then applied to prove Theorem I.

As is common with proofs involving dual quasi-bialgebras or quasi-bialgebras, some lengthy computations are involved; we have attempted to make these as transparent as possible. We outline the proof here.

Sections 2 and 3 review definitions and set the stage for later computations. Pre-bialgebras with cocycle, the objects in the category \mathcal{R} , are familiar from [AMSte], [AMStu], [ABM]; the definitions are reviewed in Subsection 2.3. As well, in Section 2, we review the idea of the splitting datum (A, H, π, σ) corresponding to a pre-bialgebra with cocycle (R, ξ) where $A = R \#_{\xi} H$ is a kind of bosonization of R and H , and remind the reader of some preliminary results from the literature. In Section 3, we review the cohomology of a bialgebra and define cohomology for a pre-bialgebra with cocycle (R, ξ) in terms of the cohomology for the K -bialgebra $A = R \#_{\xi} H$. The computations in this section simplify future calculations considerably.

For (R, ξ) in ${}^H_H\mathcal{YD}$, the definition of the multiplication map $m_R : R \otimes R \rightarrow R$ involves the cocycle $\xi : R \otimes R \rightarrow H$. In order to twist R we want to use a map $v : R \otimes R \rightarrow K$ which is related to ξ . The bijection between the set Ξ of maps $\xi : R \otimes R \rightarrow H$ so that (R, ξ) is a pre-bialgebra with cocycle and a set $V \subset \text{Hom}(R \otimes R, K)$ is key to our computations. This correspondence is established in Section 4. Essentially what happens in this section is that we show that there is an isomorphism of categories from \mathcal{R} to a category \mathcal{R}' whose objects are pairs (R, v) where R is a connected pre-bialgebra in ${}^H_H\mathcal{YD}$ and $v \in V \subset \text{Hom}(R \otimes R, K)$, with V corresponding to Ξ . Morphisms in this category are multiplicative unitary coalgebra homomorphisms $f : R \rightarrow S$ in ${}^H_H\mathcal{YD}$ such that $v_S \circ (f \otimes f) = v_R$. Then a twisting functor takes \mathcal{R}' to \mathcal{Q} and T_1 is the composite of these.

In Section 5, dual quasi-bialgebras are reviewed, dual quasi-bialgebras in a Yetter-Drinfeld category are introduced and we show that there is a process of bosonization taking dual quasi-bialgebras in ${}^H_H\mathcal{YD}$ to dual quasi-bialgebras in vector spaces. The language of cohomology from Section 3.2 is used throughout these computations. In Subsection 5.3 we show that Diagram (1) commutes. The proof of the main theorem in Section 6 follows.

2. PRELIMINARIES

Throughout we work over a field K of characteristic 0, and all maps are assumed to be K -linear. We will use Sweedler-Heyneman notation for the comultiplication in a K -coalgebra C but with the summation sign omitted, namely $\Delta(x) = x_{(1)} \otimes x_{(2)}$ for $x \in C$. For C a coalgebra and A an algebra the convolution multiplication in $\text{Hom}(C, A)$ will be denoted $*$. Composition of functions will be denoted by \circ or by juxtaposition when the meaning is clear.

For a Hopf algebra H , and M, N left H -modules, we will denote by $\text{Hom}_H(M, N)$ the left H -linear maps from M to N . If M, N are H -bimodules, then $\text{Hom}_{H,H}$ will denote the H -bimodule maps from M to N . If M, N are left H -comodules, then $\text{Hom}^H(M, N)$ will denote the left H -colinear maps from M to N . If M, N are H -bicomodules, then the bicomodule morphisms from M to N will be denoted by $\text{Hom}^{H,H}(M, N)$.

If C is a coalgebra and A an algebra, $\text{Reg}(C, A)$ denotes the convolution invertible maps from C to A . If C is a left H -module coalgebra and A a left H -module algebra, then the convolution product of left H -linear maps from C to A is also left H -linear so that $\text{Hom}_H(C, A)$ is a submonoid of $\text{Hom}(C, A)$. The notation $\text{Reg}_H(C, A)$ will denote the convolution invertible left H -linear maps from C to A . Similarly we define $\text{Reg}_{H,H}(C, A)$, $\text{Reg}^H(C, A)$, etc.

A Hopf algebra H is a left H -module under the adjoint action $h \rightarrow m = h_{(1)}mS_H(h_{(2)})$ and has a similar right adjoint action. The symbol \rightarrow may be omitted when the context makes the meaning clear. Recall [AMSte, Definition 2.7] that a left and right integral $\lambda \in H^*$ for H is called ad-invariant if $\lambda(1) = 1$ and λ is a left and right H -module map with respect to the left and right adjoint actions i.e., for all $h, x \in H$, one has

$$\lambda[h_{(1)}xS_H(h_{(2)})] = \varepsilon_H(h)\lambda(x), \quad \lambda[S_H(h_{(1)})xh_{(2)}] = \varepsilon_H(h)\lambda(x).$$

If H is semisimple and cosemisimple (for example, since characteristic zero is assumed, if H is finite dimensional cosemisimple), then the total integral for H is ad-invariant; see either [SvO, Proposition 1.12, b)] or [AMSte, Theorem 2.27]. If H has an ad-invariant integral, then H is cosemisimple.

We point out that the group algebra KG , which is in general not semisimple when K has positive characteristic, always admits an ad -invariant integral, namely the map λ defined on G by setting $\lambda(g) = \delta_{g,e}$ (Kronecker delta) and extended by linearity.

Throughout this paper we will assume that H is a K -Hopf algebra with an ad -invariant integral λ . Note that in this case H is cosemisimple whence it has bijective antipode, see [La, Theorem 3.3].

We assume familiarity with the general theory of Hopf algebras; good references are [Sw], [Mo].

2.1. The category ${}^H_H\mathcal{YD}$. Let $(\mathcal{M}, \otimes, \mathbf{1})$ be a monoidal category. A coaugmented coalgebra (C, u_C) consists of a coalgebra C in \mathcal{M} and a coalgebra homomorphism $u_C : \mathbf{1} \rightarrow C$ in \mathcal{M} , called the coaugmentation map. In particular, a coaugmented coalgebra in the category of vector spaces over a field K is a K -coalgebra C and a K -linear map $u_C : K \rightarrow C$, such that $\varepsilon(1_C) = 1_K$ and $\Delta(1_C) = 1_C \otimes 1_C$ where $1_C := u_C(1_K)$. A coaugmented coalgebra C is called connected if $C_0 = K1_C$.

Coalgebras in ${}^H_H\mathcal{YD}$, the category of left-left Yetter-Drinfeld modules over H , will play a central role in this paper. For (V, \cdot) a left H -module, we write hv for $h \cdot v$, the action of h on v , if the meaning is clear. The left H -module H with the left adjoint action is denoted (H, \rightharpoonup) ; the left and right actions of H on H induced by multiplication will be denoted by juxtaposition. For (V, ρ) a left H -comodule, we write $\rho(v) = v_{(-1)} \otimes v_{(0)}$ for the coaction. Recall that if V is a left H -module and a left H -comodule, then V is an object in ${}^H_H\mathcal{YD}$ if

$$\rho(h \cdot v) = h_{(1)}v_{(-1)}S_H(h_{(3)}) \otimes h_{(2)} \cdot v_{(0)}.$$

For example, $(H, \rightharpoonup, \Delta_H)$ is a Yetter-Drinfeld module. The category ${}^H_H\mathcal{YD}$ is braided monoidal. The tensor product of two Yetter-Drinfeld modules V and W is a Yetter-Drinfeld module with structures $h(v \otimes w) = h_{(1)}v \otimes h_{(2)}w$ and $\rho(v \otimes w) = v_{(-1)}w_{(-1)} \otimes v_{(0)} \otimes w_{(0)}$. The unit object is K with trivial structures i.e. $hk = \varepsilon_H(h)k$ and $\rho(k) = 1_H \otimes k$. The braiding $c_{V,W} : V \otimes W \rightarrow W \otimes V$ is given by $c_{V,W}(v \otimes w) = v_{(-1)}w \otimes v_{(0)}$ with inverse $c_{V,W}^{-1}(w \otimes v) = v_{(0)} \otimes S_H^{-1}(v_{(-1)})w$.

For C a coalgebra in ${}^H_H\mathcal{YD}$, we use a modified version of the Sweedler notation, writing superscripts instead of subscripts, so that comultiplication is written

$$\Delta_C(x) = \Delta(x) = x^{(1)} \otimes x^{(2)}, \text{ for every } x \in C.$$

For C a coaugmented coalgebra in ${}^H_H\mathcal{YD}$, then u_C is also required to be a map in the Yetter-Drinfeld category, i.e.,

$$h \cdot 1_C = \varepsilon_H(h)1_C \quad \text{and} \quad \rho_C(1_C) = 1_H \otimes 1_C.$$

If C and D are coalgebras in ${}^H_H\mathcal{YD}$, so is $C \otimes D$ with Yetter-Drinfeld module structure given as above, counit $\varepsilon_{C \otimes D} = \varepsilon_C \otimes \varepsilon_D$ and $\Delta_{C \otimes D} = (C \otimes c_{C,D} \otimes D) \circ (\Delta_C \otimes \Delta_D)$, so that

$$\Delta_{C \otimes D}(x \otimes y) = x^{(1)} \otimes x_{(-1)}^{(2)} y^{(1)} \otimes x_{(0)}^{(2)} \otimes y^{(2)}.$$

If C, D are coaugmented so is $C \otimes D$ with $1_{C \otimes D} = 1_C \otimes 1_D$.

Given three coalgebras C, D, E in ${}^H_H\mathcal{YD}$ the coalgebras $(C \otimes D) \otimes E$ and $C \otimes (D \otimes E)$ are isomorphic and can both be denoted by $C \otimes D \otimes E$.

For $(C, \Delta_C, \varepsilon_C)$ a left H -comodule coalgebra, Δ_C and ε_C are morphisms of left H -comodules i.e. for all $c \in C$,

$$(2) \quad c_{(-1)} \otimes (c_{(0)})^{(1)} \otimes (c_{(0)})^{(2)} = (c^{(1)})_{(-1)}(c^{(2)})_{(-1)} \otimes (c^{(1)})_{(0)} \otimes (c^{(2)})_{(0)} \text{ and}$$

$$(3) \quad c_{(-1)}\varepsilon_C(c_{(0)}) = 1_H\varepsilon_C(c).$$

Similarly, for $(C, \Delta_C, \varepsilon_C)$ a left H -module coalgebra, Δ_C and ε_C are morphisms of left H -modules i.e. for all $h \in H, c \in C$,

$$(4) \quad (hc)^{(1)} \otimes (hc)^{(2)} = h_{(1)}c^{(1)} \otimes h_{(2)}c^{(2)} \text{ and}$$

$$(5) \quad \varepsilon_C(hc) = \varepsilon_H(h)\varepsilon_C(c).$$

For a left-left Yetter-Drinfeld module C , $(C, \Delta_C, \varepsilon_C)$ is a coalgebra in ${}^H_H\mathcal{YD}$ if $(C, \Delta_C, \varepsilon_C)$ is both a left H -comodule coalgebra and a left H -module coalgebra.

LEMMA 2.1. *Let C_1, D_1 be left H -comodule coalgebras and let C_2, D_2 be left H -module coalgebras. Then $C := C_1 \otimes C_2$ is a coalgebra via*

$$\Delta_C(z \otimes w) = z^{(1)} \otimes z_{(-1)}^{(2)} w^{(1)} \otimes z_{(0)}^{(2)} \otimes w^{(2)} \quad \text{and} \quad \varepsilon_C(z \otimes w) = \varepsilon_{C_1}(z) \varepsilon_{C_2}(w);$$

$D := D_1 \otimes D_2$ is a coalgebra in the same way. Let $f : C_1 \rightarrow D_1$ be a morphism of left H -comodule coalgebras and let $g : C_2 \rightarrow D_2$ be a morphism of left H -module coalgebras. Then $f \otimes g : C \rightarrow D$ is a coalgebra map.

Proof. We must show that $(C, \Delta_C, \varepsilon_C)$ is a coalgebra. The comultiplication map Δ_C is coassociative since for $y = z \otimes w$ with $z \in C_1, w \in C_2$,

$$\begin{aligned} (C \otimes \Delta) \Delta(y) &= z^{(1)} \otimes z_{(-1)}^{(2)} w^{(1)} \otimes \Delta(z_{(0)}^{(2)} \otimes w^{(2)}) \\ &= z^{(1)} \otimes z_{(-1)}^{(2)} w^{(1)} \otimes \left(z_{(0)}^{(2)}\right)^{(1)} \otimes \left(z_{(0)}^{(2)}\right)^{(2)}_{(-1)} w^{(2)} \otimes \left(z_{(0)}^{(2)}\right)^{(2)}_{(0)} \otimes w^{(3)} \\ &\stackrel{(2)}{=} z^{(1)} \otimes z_{(-1)}^{(2)} z_{(-1)}^{(3)} w^{(1)} \otimes z_{(0)}^{(2)} \otimes \left(z_{(0)}^{(3)}\right)_{(-1)} w^{(2)} \otimes \left(z_{(0)}^{(3)}\right)_{(0)} \otimes w^{(3)} \\ &= z^{(1)} \otimes z_{(-1)}^{(2)} z_{(-2)}^{(3)} w^{(1)} \otimes z_{(0)}^{(2)} \otimes z_{(-1)}^{(3)} w^{(2)} \otimes z_{(0)}^{(3)} \otimes w^{(3)} \\ &\stackrel{(4)}{=} z^{(1)} \otimes z_{(-1)}^{(2)} \left(z_{(-1)}^{(3)} w^{(1)}\right)^{(1)} \otimes z_{(0)}^{(2)} \otimes \left(z_{(-1)}^{(3)} w^{(1)}\right)^{(2)} \otimes z_{(0)}^{(3)} \otimes w^{(2)} \\ &= \left(z^{(1)}\right)^{(1)} \otimes \left(z^{(1)}\right)^{(2)}_{(-1)} \left(z_{(-1)}^{(2)} w^{(1)}\right)^{(1)} \otimes \left(z^{(1)}\right)^{(2)}_{(0)} \otimes \left(z_{(-1)}^{(2)} w^{(1)}\right)^{(2)} \otimes z_{(0)}^{(2)} \otimes w^{(2)} \\ &= \Delta\left(z^{(1)} \otimes z_{(-1)}^{(2)} w^{(1)}\right) \otimes z_{(0)}^{(2)} \otimes w^{(2)} = (\Delta \otimes C) \Delta(y). \end{aligned}$$

Also, applying $C \otimes \varepsilon_C$ to $\Delta(y)$, we obtain

$$\begin{aligned} (C \otimes \varepsilon_C) \Delta(y) &= z^{(1)} \otimes z_{(-1)}^{(2)} w^{(1)} \varepsilon_C\left(z_{(0)}^{(2)} \otimes w^{(2)}\right) = z^{(1)} \otimes z_{(-1)}^{(2)} w^{(1)} \varepsilon_{C_1}\left(z_{(0)}^{(2)}\right) \varepsilon_{C_2}\left(w^{(2)}\right) \\ &= z^{(1)} \otimes z_{(-1)}^{(2)} \varepsilon_{C_1}\left(z_{(0)}^{(2)}\right) w \stackrel{(3)}{=} y, \end{aligned}$$

and similarly, using (5), we obtain $(\varepsilon_C \otimes C) \Delta_C(y) = y$. The final statement is clear since f, g preserve the relevant coalgebra, H -module and H -comodule structures. \square

REMARK 2.2. For C_1, C_2, D_1, D_2, f, g as in Lemma 2.1, since $f \otimes g$ is a coalgebra map, then, for A an algebra and maps $\alpha, \beta : D \rightarrow A$, we have

$$(6) \quad [\alpha \circ (f \otimes g)] * [\beta \circ (f \otimes g)] = (\alpha * \beta) \circ (f \otimes g).$$

2.2. Some preliminary results. We recall some key definitions and results from [ABM].

DEFINITION 2.3. [ABM, Definition 2.1] For M a left H -comodule, define $\Psi : \text{Hom}(M, K) \rightarrow \text{Hom}^H(M, H)$ by

$$\Psi(\alpha) = (H \otimes \alpha) \rho_M.$$

REMARK 2.4. (i) [ABM, Remark 2.2] Let $f : M \rightarrow L$ be a morphism of left H -comodules and $\alpha \in \text{Hom}(L, K)$. Then $\Psi(\alpha) \circ f = \Psi(\alpha \circ f)$.

(ii) [ABM, Lemma 2.3] For C a left H -comodule coalgebra, $\Psi : \text{Hom}(C, K) \rightarrow \text{Hom}^H(C, H)$ is an algebra isomorphism. The inverse Ψ^{-1} is defined by $\Psi^{-1}(\alpha) = \varepsilon_H \alpha$.

The next lemma notes that Ψ, Ψ^{-1} in the preceding remark are isomorphisms between the subalgebras $\text{Hom}_H(C, K)$ of $\text{Hom}(C, K)$ where K has the trivial H -action and $\text{Hom}_H(C, H)$ of $\text{Hom}(C, H)$ where H has the left adjoint H -action.

LEMMA 2.5. *For C a coalgebra in ${}^H_H\mathcal{YD}$, $\Psi : \text{Hom}_H(C, K) \rightarrow {}^H_H\mathcal{YD}(C, H)$ is an algebra isomorphism.*

Proof. Let $\alpha \in \text{Hom}_H(C, K)$ and we check that $\Psi(\alpha)$ is left H -linear.

$$\begin{aligned} \Psi(\alpha)(hc) &= (H \otimes \alpha) \rho_M(hc) = (hc)_{\langle -1 \rangle} \alpha((hc)_{\langle 0 \rangle}) \\ &= h_{(1)} c_{\langle -1 \rangle} S_H(h_{(3)}) \alpha[h_{(2)} c_{\langle 0 \rangle}] \\ &= h_{(1)} c_{\langle -1 \rangle} S_H(h_{(3)}) \varepsilon_H(h_{(2)}) \alpha(c_{\langle 0 \rangle}) \\ &= h_{(1)} c_{\langle -1 \rangle} \alpha(c_{\langle 0 \rangle}) S_H(h_{(2)}) \\ &= h_{(1)} [\Psi(\alpha)(c)] S_H(h_{(2)}) = [h \rightharpoonup \Psi(\alpha)(c)]. \end{aligned}$$

Similarly if β is an H -linear map from C to H , then

$$\Psi^{-1}(\beta)(hc) = \varepsilon_H \beta(hc) = \varepsilon_H(h) \varepsilon_H(\beta(c)) = h \Phi^{-1}(\beta)(c).$$

□

The previous result depends on the fact that the forgetful functor from ${}^H_H\mathcal{YD}$ to the category of left H -modules is left adjoint to the functor $H \otimes (-)$, see e.g. [Ar, Claim 3.3].

DEFINITION 2.6. For C a coalgebra in ${}^H_H\mathcal{YD}$ and $\alpha \in \text{Hom}(C, H)$, α is called a *normalized dual Sweedler 1-cocycle* if $\Delta_H \alpha = (m_H \otimes \alpha)(\alpha \otimes \rho_C) \Delta_C$ and $\varepsilon_H \alpha = \varepsilon_C$. Equivalently, for $x \in C$,

$$(7) \quad \alpha(x)_{(1)} \otimes \alpha(x)_{(2)} = \alpha(x^{(1)}) x_{\langle -1 \rangle}^{(2)} \otimes \alpha(x_{\langle 0 \rangle}^{(2)}) \quad \text{and} \quad \varepsilon_H(\alpha(x)) = \varepsilon_C(x).$$

REMARK 2.7. Any $\alpha : C \rightarrow H$ satisfying (7) above is convolution invertible by [ABM, Proposition 2.6].

LEMMA 2.8. [ABM, Lemma 2.7] *Let C be a coalgebra and let (M, μ) be a left H -module. Define*

$$\Phi : \text{Hom}(C, H) \rightarrow \text{End}(C \otimes M) \text{ by } \Phi(\alpha) := (C \otimes \mu_M) \circ [(C \otimes \alpha) \Delta_C \otimes M],$$

for $\alpha \in \text{Hom}(C, H)$. The map Φ is an algebra homomorphism.

The following observation is from [ABM]:

REMARK 2.9. [ABM, Remark 2.4]

(i) For C a left H -module coalgebra and $v \in \text{Reg}(C, K)$, then v is left H -linear if and only if v^{-1} is.

(ii) A result of Takeuchi (see [Mo, Lemma 5.2.10]) shows that if $(C, \Delta_C, \varepsilon_C, u_C)$ is a coaugmented connected coalgebra, then every $v : C \rightarrow K$ such that $v(1_C) = 1_K$ is convolution invertible and $v^{-1}(1_C) = 1_K$ also.

LEMMA 2.10. *Let C be a left H -comodule coalgebra, C' a left H -module coalgebra and let $C \otimes C'$ have the coalgebra structure from Lemma 2.1. Let $u, v \in \text{Hom}(C, K)$ with u convolution invertible and let $u', v' \in \text{Hom}_H(C', K)$. For vector spaces D, D' , let $f : C \rightarrow D$ and $g : C' \rightarrow D'$ and for a K -algebra W , let $\alpha : D \otimes D' \rightarrow W$. Then*

$$(8) \quad (u \otimes u') * [\alpha \circ (f \otimes g)] * (v \otimes v') = \alpha \circ (f^u \otimes u' * g * v') \circ \Phi[u * \Psi(v)]$$

where $f^u := u * f * u^{-1}$. In particular, for $D' := C'$, one has

$$(9) \quad (u \otimes \varepsilon_{C'}) * [\alpha \circ (f \otimes C')] = \alpha \circ (u * f \otimes C')$$

If v is also left H -colinear, we have

$$(10) \quad [\alpha \circ (f \otimes C')] * (v \otimes \varepsilon_{C'}) = \alpha \circ (f * v \otimes C')$$

Proof. For every $z \in C$ and $t \in C'$, we have

$$\begin{aligned} & \alpha(f^u \otimes u' * g * v') \Phi[u * \Psi(v)](z \otimes t) \\ &= \alpha(f^u \otimes u' * g * v') \left(z^{(1)} \otimes [u * \Psi(v)](z^{(2)}) t \right) \\ &= \alpha(\text{Id} \otimes u' * g * v') \left(f^u(z^{(1)}) \otimes u(z^{(2)}) z_{\langle -1 \rangle}^{(3)} v(z_{\langle 0 \rangle}^{(3)}) t \right) \\ &= \alpha(\text{Id} \otimes u' * g * v') \left(f^u(z^{(1)}) u(z^{(2)}) \otimes z_{\langle -1 \rangle}^{(3)} t \right) v \left(z_{\langle 0 \rangle}^{(3)} \right) \end{aligned}$$

$$\begin{aligned}
&= \alpha(\text{Id} \otimes u' * g * v') \left(u(z^{(1)})f(z^{(2)}) \otimes z_{(-1)}^{(3)}t \right) v \left(z_{(0)}^{(3)} \right) \\
&= u(z^{(1)})\alpha(\text{Id} \otimes u' * g * v') \left(f(z^{(2)}) \otimes z_{(-1)}^{(3)}t \right) v \left(z_{(0)}^{(3)} \right) \\
&= u(z^{(1)})\alpha \left(f(z^{(2)}) \otimes (u' * g * v') \left(z_{(-1)}^{(3)}t \right) \right) v \left(z_{(0)}^{(3)} \right) \\
&\stackrel{(4)}{=} u(z^{(1)})u' \left(z_{(-3)}^{(3)}t^{(1)} \right) \alpha \left(f(z^{(2)}) \otimes g \left(z_{(-2)}^{(3)}t^{(2)} \right) \right) v \left(z_{(0)}^{(3)} \right) v' \left(z_{(-1)}^{(3)}t^{(3)} \right) \\
&= u(z^{(1)})u' \left(t^{(1)} \right) \alpha \left(f(z^{(2)}) \otimes g \left(z_{(-1)}^{(3)}t^{(2)} \right) \right) v \left(z_{(0)}^{(3)} \right) v' \left(t^{(3)} \right) \text{ by } u', v' \text{ left } H\text{-linear} \\
&= u(z^{(1)})u' \left(z_{(-1)}^{(2)}z_{(-2)}^{(3)}t^{(1)} \right) \alpha(f \otimes g) \left(z_{(0)}^{(2)} \otimes z_{(-1)}^{(3)}t^{(2)} \right) v \left(z_{(0)}^{(3)} \right) v' \left(t^{(3)} \right) \text{ by } u' \text{ left } H\text{-linear} \\
&= (u \otimes u') \left(z^{(1)} \otimes z_{(-1)}^{(2)}z_{(-2)}^{(3)}t^{(1)} \right) \alpha(f \otimes g) \left(z_{(0)}^{(2)} \otimes z_{(-1)}^{(3)}t^{(2)} \right) (v \otimes v') \left(z_{(0)}^{(3)} \otimes t^{(3)} \right) \\
&= [(u \otimes u') * \alpha(f \otimes g) * (v \otimes v')] (z \otimes t),
\end{aligned}$$

and (8) is proved. It is then easy to see that (9) holds by using (8) with $u' = \varepsilon_{C'}$, $g = C'$, $v = \varepsilon_C$ and $v' = \varepsilon_{C'}$.

If v is also left H -colinear, we get

$$\begin{aligned}
&[\alpha(f \otimes C') * (v \otimes \varepsilon_{C'})] (z \otimes t) \\
&\stackrel{(8)}{=} \alpha(f \otimes C')\Phi[\Psi(v)](z \otimes t) = \alpha(f \otimes C')(z_1 \otimes [\Psi(v)](z_2)t) \\
&= \alpha(f \otimes C')(z_1 \otimes (z_2)_{-1}v[(z_2)_0]t) = \alpha(f(z_1) \otimes (z_2)_{-1}v[(z_2)_0]t) \\
&= \alpha(f(z_{(1)}) \otimes v(z_{(2)})t) = \alpha(f(z_{(1)})v(z_{(2)}) \otimes t) = [\alpha(f * v \otimes C')](z \otimes t),
\end{aligned}$$

so that (10) holds. \square

2.3. Pre-bialgebras with cocycle. Following [AMStu, Definition 2.3, Definitions 3.1], we define:

DEFINITION 2.11. A pre-bialgebra $R = (R, m_R, u_R, \Delta_R, \varepsilon_R)$ in ${}^H_H\mathcal{YD}$ is a coaugmented coalgebra $(R, \Delta_R, \varepsilon_R, u_R)$ in the category ${}^H_H\mathcal{YD}$ together with a left H -linear map $m_R : R \otimes R \rightarrow R$ such that m_R is a coalgebra homomorphism, i.e.,

$$\Delta_R m_R = (m_R \otimes m_R) \Delta_{R \otimes R} \quad \text{and} \quad \varepsilon_R m_R = m_K(\varepsilon_R \otimes \varepsilon_R),$$

and u_R is a unit for m_R , i.e.,

$$(11) \quad m_R(R \otimes u_R) = R = m_R(u_R \otimes R).$$

If clear from the context, the subscript R on the structure maps may be omitted.

Essentially a pre-bialgebra differs from a bialgebra in ${}^H_H\mathcal{YD}$ in that **the multiplication need not be associative and need not be a morphism of H -comodules.**

DEFINITION 2.12. A cocycle for the pre-bialgebra $(R, m, u, \Delta, \varepsilon)$ in ${}^H_H\mathcal{YD}$ is a map $\xi : R \otimes R \rightarrow H$ such that:

- (12) ξ is left H -linear with respect to the left H -adjoint action on H ;
- (13) $\Delta_H \xi = (m_H \otimes \xi)(\xi \otimes \rho_{R \otimes R}) \Delta_{R \otimes R}$ and $\varepsilon_H \xi = m_K(\varepsilon \otimes \varepsilon)$;
- (14) $c_{R,H}(m_R \otimes \xi) \Delta_{R \otimes R} = (m_H \otimes m_R)(\xi \otimes \rho_{R \otimes R}) \Delta_{R \otimes R}$;
- (15) $m_R(R \otimes m_R) = m_R(m_R \otimes R) \Phi(\xi)$;
- (16) $m_H(\xi \otimes H)[R \otimes (m_R \otimes \xi) \Delta_{R \otimes R}] = m_H(\xi \otimes H)(R \otimes c_{H,R})[(m_R \otimes \xi) \Delta_{R \otimes R} \otimes R]$;
- (17) ξ is unital, i.e., $\xi(R \otimes u) = \xi(u \otimes R) = \varepsilon 1_H$.

Then $(R, m, u, \Delta, \varepsilon, \xi)$, written as (R, ξ) unless more detail is needed, is called a pre-bialgebra with cocycle in ${}^H_H\mathcal{YD}$.

Note that (13) says that ξ is a normalized dual Sweedler 1-cocycle as in Definition 2.6. By Remark 2.7, ξ is convolution invertible. Condition (14) has a Yetter-Drinfeld-like form as follows.

LEMMA 2.13. *Let R be a pre-bialgebra in ${}^H_H\mathcal{YD}$, and let $\xi : R \otimes R \rightarrow H$ be a convolution invertible map. Write $C := R \otimes R$, and $m = m_R$. Then (14) holds if and only if, for all $z \in C$:*

$$(18) \quad \rho_R(m(z)) = m(z)_{\langle -1 \rangle} \otimes m(z)_{\langle 0 \rangle} = \xi(z^{(1)})z_{\langle -1 \rangle}^{(2)}\xi^{-1}(z^{(3)}) \otimes m(z_{\langle 0 \rangle}^{(2)})$$

Proof. Applying (14) to $z \in C$, yields $m(z^{(1)})_{\langle -1 \rangle}\xi(z^{(2)}) \otimes m(z^{(1)})_{\langle 0 \rangle} = \xi(z^{(1)})z_{\langle -1 \rangle}^{(2)} \otimes m(z_{\langle 0 \rangle}^{(2)})$. Thus if (14) holds, then $m(z^{(1)})_{\langle -1 \rangle}\xi(z^{(2)}) \otimes m(z^{(1)})_{\langle 0 \rangle} \otimes z^{(3)} = \xi(z^{(1)})z_{\langle -1 \rangle}^{(2)} \otimes m(z_{\langle 0 \rangle}^{(2)}) \otimes z^{(3)}$, and applying $(m_H \otimes R) \circ (H \otimes \tau) \circ (H \otimes R \otimes \xi^{-1})$ to both sides of this equation where τ is the usual twist map, we obtain (18). The argument that (18) implies (14) is similar. \square

Condition (15) describes associativity of the multiplication m_R ; it is shown in [ABM, Remark 2.11] that if $\xi(z)t = \varepsilon_{R \otimes R}(z)t$ for all $z \in R \otimes R, t \in R$ or, equivalently, if $\Phi(\xi) = Id_{R^{\otimes 3}}$ then m_R is associative. By [ABM, Theorem 3.7] the converse holds if R is connected.

2.4. Splitting data. There is a correspondence between pre-bialgebras with cocycle (R, ξ) in ${}^H_H\mathcal{YD}$ and the 4-tuples (A, H, π, σ) known as splitting data.

DEFINITION 2.14. A **splitting datum** (A, H, π, σ) consists of a bialgebra A , a Hopf algebra H , a bialgebra homomorphism $\sigma : H \rightarrow A$ and an H -bilinear coalgebra homomorphism $\pi : A \rightarrow H$ such that $\pi\sigma = Id_H$. Note that H -bilinear here means $\pi(\sigma(h)x\sigma(h')) = h\pi(x)h'$ for all $h, h' \in H$ and $x \in A$. We say that a splitting datum is **trivial** whenever π is a bialgebra homomorphism.

Given (R, ξ) , a splitting datum is constructed as follows. Let $A := R \#_{\xi} H$ have coalgebra structure equal to the smash coproduct $R \# H$ of R by H , i.e., the coalgebra defined on $R \otimes H$ by setting, for every $r \in R$ and $h \in H$,

$$(19) \quad \Delta_{R \# H}(r \# h) = r^{(1)} \# r_{\langle -1 \rangle}^{(2)} h_{(1)} \otimes r_{\langle 0 \rangle}^{(2)} \# h_{(2)}, \quad \varepsilon_{R \# H}(r \# h) = \varepsilon_R(r) \varepsilon_H(h).$$

The algebra structures are as follows. The unit $u_A(1) := 1_R \# 1_H$ and multiplication is given by

$$m_A = (R \otimes m_H) [(m_R \otimes \xi) \Delta_{R \otimes R} \otimes m_H] (R \otimes c_{H, R} \otimes H)$$

so that for $r, s \in R, h, l \in H$,

$$(20) \quad m_A(r \# h \otimes s \# l) = m_R \left(r^{(1)} \otimes r_{\langle -1 \rangle}^{(2)} h_{(1)} s^{(1)} \right) \# \xi \left(r_{\langle 0 \rangle}^{(2)} \otimes h_{(2)} s^{(2)} \right) h_{(3)} l.$$

Unless $\xi(R \otimes R) = K$, the action of $\xi(R \otimes R)$ will not be trivial. It is useful to note that:

$$(21) \quad (R \otimes \varepsilon_H) m_A(r \# h \otimes s \# l) = m_R(r \otimes h s) \varepsilon_H(l), \quad (\varepsilon_R \otimes H) m_A(r \# h \otimes s \# l) = \xi(r \otimes h_{(1)} s) h_{(2)} l.$$

Note that the canonical injection $\sigma : H \hookrightarrow R \#_{\xi} H$ is a bialgebra homomorphism. Furthermore

$$\pi : R \#_{\xi} H \rightarrow H : r \# h \mapsto \varepsilon(r) h$$

is an H -bilinear coalgebra retraction of σ .

Conversely, suppose that (A, H, π, σ) is a splitting datum and we find $(R, m, u, \Delta, \varepsilon, \xi)$, the associated pre-bialgebra with cocycle in ${}^H_H\mathcal{YD}$ as in [ABM, 2.2.3]. As when π is a bialgebra morphism and A is a Radford biproduct, set

$$R = A^{c \circ \pi} = \{a \in A \mid a_{(1)} \otimes \pi(a_{(2)}) = a \otimes 1_H\},$$

and let

$$\tau : A \rightarrow R, \quad \tau(a) = a_{(1)} \sigma S_H \pi(a_{(2)}).$$

Define a left-left Yetter-Drinfeld structure on R by

$$h \cdot r = hr = \sigma(h_{(1)}) r \sigma S_H(h_{(1)}), \quad \rho(r) = \pi(r_{(1)}) \otimes r_{(2)},$$

and define a coalgebra structure in ${}^H_H\mathcal{YD}$ on R by

$$(22) \quad \Delta(r) = r^{(1)} \otimes r^{(2)} = r_{(1)} \sigma S_H \pi(r_{(2)}) \otimes r_{(3)} = \tau(r_{(1)}) \otimes r_{(2)}, \quad \varepsilon = \varepsilon_{A|_R}.$$

The map

$$\omega : R \otimes H \rightarrow A, \quad \omega(r \otimes h) = r \sigma(h)$$

is an isomorphism of K -vector spaces, the inverse being defined by

$$\omega^{-1} : A \rightarrow R \otimes H, \quad \omega^{-1}(a) = a_{(1)}\sigma S_H\pi(a_{(2)}) \otimes \pi(a_{(3)}) = \tau(a_{(1)}) \otimes \pi(a_{(2)}).$$

Clearly A defines, via ω , a bialgebra structure on $R \otimes H$ that will depend on σ and π . As shown in [Scha, 6.1] and [AMSte, Theorem 3.64], $(R, m, u, \Delta, \varepsilon)$ is a pre-bialgebra in ${}^H_H\mathcal{YD}$ with cocycle ξ where the maps $u : K \rightarrow R$ and $m : R \otimes R \rightarrow R$, are defined by

$$u = u_A^{|R|}, \quad m(r \otimes s) = r_{(1)}s_{(1)}\sigma S_H\pi(r_{(2)}s_{(2)}) = \tau(r \cdot_A s)$$

and the cocycle $\xi : R \otimes R \rightarrow H$ is the map defined by $\xi(r \otimes s) = \pi(r \cdot_A s)$. Then (R, ξ) is the pre-bialgebra with cocycle in ${}^H_H\mathcal{YD}$ associated to (A, H, π, σ) . Moreover, $\omega : R \#_\xi H \rightarrow A$ is a bialgebra isomorphism.

2.5. Monoids of H -bilinear multibalanced and H -linear maps. Let C be a coalgebra in ${}^H_H\mathcal{YD}$ and set $A := C \# H$, the smash coproduct of C by H . We point out (cf. [AMSte, Example 3.17]) that A becomes a coalgebra in the monoidal category $({}^H_H\mathcal{M}_H^H, \square_H, H)$.

It is clear that we can regard $A^{\otimes n}$ as an H -bimodule via the structures of the first left (resp. right) hand-side factor. Regard $C^{\otimes n}$ as a left H -module via the diagonal action. For $n > 1$, a map $f : A^{\otimes n} \rightarrow K$ is called H -multibalanced if for all $a^1, \dots, a^n \in A, h \in H$, one has $f(a^1 \otimes \dots \otimes a^i h \otimes a^{i+1} \otimes \dots \otimes a^n) = f(a^1 \otimes \dots \otimes a^i \otimes h a^{i+1} \otimes \dots \otimes a^n)$ for $1 \leq i \leq n-1$. Sets of multibalanced maps will be denoted by a superscript b . For example

$$\text{Hom}_{H,H}^b(A^{\otimes n}, K) = \{f | f \in \text{Hom}_{H,H}(A^{\otimes n}, K) \text{ and, if } n > 1, f \text{ is } H\text{-multibalanced}\}$$

is the submonoid of $\text{Hom}_{H,H}(A^{\otimes n}, K)$ of H -multibalanced maps.

LEMMA 2.15. *For $n \in \mathbb{N}$, there is an isomorphism of monoids*

$$\Omega^n = \Omega_{H,C}^n : (\text{Hom}_{H,H}^b(A^{\otimes n}, K), *, \varepsilon_{A^{\otimes n}}) \rightarrow (\text{Hom}_H(C^{\otimes n}, K), *, \varepsilon_{C^{\otimes n}})$$

defined by $\gamma \mapsto \Omega^n(\gamma)$ where

$$\Omega^n(\gamma)(c^1 \otimes c^2 \otimes \dots \otimes c^{n-1} \otimes c^n) := \gamma((c^1 \# 1_H) \otimes (c^2 \# 1_H) \otimes \dots \otimes (c^{n-1} \# 1_H) \otimes (c^n \# 1_H)),$$

with inverse $\mathcal{U}^n = \mathcal{U}_{H,C}^n$ given by $v \mapsto \mathcal{U}^n(v)$ where

$$\begin{aligned} \mathcal{U}^n(v) & ((c^1 \# h^1) \otimes (c^2 \# h^2) \otimes \dots \otimes (c^{n-1} \# h^{n-1}) \otimes (c^n \# h^n)) \\ & := v(c^1 \otimes h_{(1)}^1 c^2 \otimes \dots \otimes h_{(n-2)}^1 \dots h_{(2)}^{n-3} h_{(1)}^{n-2} c^{n-1} \otimes h_{(n-1)}^1 \dots h_{(3)}^{n-3} h_{(2)}^{n-2} h^{n-1} c^n) \varepsilon_H(h^n). \end{aligned}$$

Proof. The proof is straightforward, cf. [ABM, Proposition 4.9]. \square

Clearly the isomorphism Ω^n induces an isomorphism

$$\Omega_{H,C}^n : \text{Reg}_{H,H}^b(A^{\otimes n}, K) \rightarrow \text{Reg}_H(C^{\otimes n}, K)$$

with inverse $\mathcal{U}_{H,C}^n$.

Assume furthermore that C is coaugmented and call a map ϕ in $\text{Hom}(A^{\otimes n}, K)$ or in $\text{Hom}(C^{\otimes n}, K)$ unital if $\phi = \varepsilon$ on elements of the form $x^1 \otimes \dots \otimes x^n$ with at least one of the x^i equal to 1. If ϕ is unital and convolution invertible, then ϕ^{-1} is also unital. It is easy to see that Ω^n and \mathcal{U}^n preserve unitality.

3. COHOMOLOGY OF (PRE-)BIALGEBRAS

3.1. Cohomology of a K -bialgebra. Recall (cf. [Ka, dual to page 368]) that a coalgebra with multiplication and unit is a datum $(E, m, u, \Delta, \varepsilon)$ where (E, Δ, ε) is a K -coalgebra, $m : E \otimes E \rightarrow E$ is a coalgebra homomorphism called multiplication (which may fail to be associative) and $u : K \rightarrow E$ is a coalgebra homomorphism called unit. Let $(E, m, u, \Delta, \varepsilon)$ be a coalgebra with multiplication and unit. For $t \in \mathbb{N}$ and $0 \leq i \leq t+1$ define the maps

$$m_i^{t+1} : E^{\otimes(t+1)} \rightarrow E^{\otimes t}$$

as follows. If $t = 0$ we set $m_0^1 = \varepsilon = m_1^1$ while for $t > 0$ we set

$$m_i^{t+1}(x_1 \otimes \cdots \otimes x_i \otimes \cdots \otimes x_{t+1}) = \begin{cases} \varepsilon(x_1)x_2 \otimes \cdots \otimes x_i \otimes \cdots \otimes x_{t+1} & \text{for } i = 0, \\ x_1 \otimes \cdots \otimes x_i x_{i+1} \otimes \cdots \otimes x_{t+1} & \text{for } 1 \leq i \leq t, \\ x_1 \otimes \cdots \otimes x_i \otimes \cdots \otimes x_t \varepsilon(x_{t+1}) & \text{for } i = t+1. \end{cases}$$

For $w \in \text{Reg}(E^{\otimes t}, K)$, note that $wm_i^{t+1} \in \text{Reg}(E^{\otimes(t+1)}, K)$ since m_i^{t+1} is a coalgebra map so that $wm_i^{t+1} * w^{-1}m_i^{t+1} = (w * w^{-1})m_i^{t+1} = \varepsilon_{E^{\otimes t}}m_i^{t+1} = \varepsilon_{E^{\otimes(t+1)}}$ and thus

$$w^{-1}m_i^{t+1} = (wm_i^{t+1})^{-1}.$$

For $w \in \text{Reg}(E^{\otimes t}, K)$, define (cf. [Maj, pages 60, 53]) the two elements $\partial_+^t(w)$ and $\partial_-^t(w)$ in $\text{Reg}(E^{\otimes(t+1)}, K)$ to be the convolution products:

$$(\partial_E^t)_+(w) = \partial_+^t(w) = \prod_{\substack{i=0, \dots, t+1 \\ i \text{ even}}} wm_i^{t+1}, \quad (\partial_E^t)_-(w) = \partial_-^t(w) = \prod_{\substack{i=0, \dots, t+1 \\ i \text{ odd}}} wm_i^{t+1}.$$

In particular, for $t = 0, 1, 2, 3$:

$$\begin{aligned} t = 0: \quad & \partial_+^0(w) = wm_0^1 = w\varepsilon; \quad \partial_-^0(w) = wm_1^1 = w\varepsilon, \\ t = 1: \quad & \partial_+^1(w) = wm_0^2 * wm_2^2 = m_K(\varepsilon \otimes w) * m_K(w \otimes \varepsilon) = m_K(w \otimes w), \\ & \partial_-^1(w) = wm_1^2 = wm, \\ t = 2: \quad & \partial_+^2(w) = wm_0^3 * wm_2^3 = m_K(\varepsilon \otimes w) * w(E \otimes m), \\ & \partial_-^2(w) = wm_1^3 * wm_3^3 = w(m \otimes E) * m_K(w \otimes \varepsilon), \\ t = 3: \quad & \partial_+^3(w) = wm_0^4 * wm_2^4 * wm_4^4 = m_K(\varepsilon \otimes w) * w(E \otimes m \otimes E) * m_K(w \otimes \varepsilon), \\ & \partial_-^3(w) = wm_1^4 * wm_3^4 = w(m \otimes E \otimes E) * w(E \otimes E \otimes m). \end{aligned}$$

Now define the maps:

$$\partial^t = \partial_E^t : \text{Reg}(E^{\otimes t}, K) \rightarrow \text{Reg}(E^{\otimes(t+1)}, K) : w \mapsto \partial_+^t(w) * \partial_-^t(w^{-1}),$$

and by the definition of ∂^t :

$$\begin{aligned} \partial^0(w) &= \varepsilon, \\ \partial^1(w) &= m_K(w \otimes w) * w^{-1}m, \\ \partial^2(w) &= m_K(\varepsilon \otimes w) * w(E \otimes m) * w^{-1}(m \otimes E) * m_K(w^{-1} \otimes \varepsilon), \\ \partial^3(w) &= m_K(\varepsilon \otimes w) * w(E \otimes m \otimes E) * m_K(w \otimes \varepsilon) * w^{-1}(m \otimes E \otimes E) * w^{-1}(E \otimes E \otimes m). \end{aligned}$$

By definition, a t -cocycle is an element $w \in \text{Reg}(E^{\otimes t}, K)$ such that $\partial^t(w) = \varepsilon_{E^{\otimes(t+1)}}$ so that $Z^2(A, K)$ ([Doi], [ABM, Section 4]) is just $\text{Ker}(\partial^2)$. A t -coboundary is an element in $\text{Im}(\partial^{t-1})$.

In general it is not clear whether ∂^t is a group homomorphism and any t -coboundary is a t -cocycle. This holds for some E ; for example it is true if E is a cocommutative bialgebra. In general, $\partial^1 \partial^0(w) = \varepsilon_{E \otimes E}$. Moreover, if m is associative, then $\partial^2 \partial^1(w) = \varepsilon_{E^{\otimes 3}}$, see [BC, Lemma 1].

3.2. Cohomology of a pre-bialgebra with cocycle. Let (R, ξ) be a connected pre-bialgebra with cocycle in ${}^H_H \mathcal{YD}$ with associated splitting datum $(A := R \#_\xi H, H, \pi, \sigma)$ as outlined in the preliminaries. Since A is a bialgebra, we can consider the maps

$$(\partial_A^t)_+, (\partial_A^t)_-, \partial_A^t : \text{Reg}(A^{\otimes t}, K) \rightarrow \text{Reg}(A^{\otimes(t+1)}, K),$$

where $\partial_A^t(w) = \partial_+^t(w) * \partial_-^t(w^{-1})$ as in the previous section. Since the multiplication of A is H -bilinear and H -balanced, (cf. [AMSte, Theorem 3.62]) it is clear that these induce maps

$$(\partial_A^t)_+, (\partial_A^t)_-, \partial_A^t : \text{Reg}_{H,H}^b(A^{\otimes t}, K) \rightarrow \text{Reg}_{H,H}^b(A^{\otimes(t+1)}, K).$$

Since $(R, \Delta, \varepsilon, m, u)$ is not a coalgebra with multiplication and unit over K in the sense of Subsection 3.1 (the coalgebra structure on $R \otimes R$ is different as here it depends on the braiding of ${}^H_H \mathcal{YD}$), the definition of cohomology given in the previous section does not apply to R . In order to define a suitable cohomology for $\text{Reg}(R^{\otimes t}, K)$ we use the isomorphisms Ω and \bar{U} from Section 2.5.

Note that this is the same approach taken in [ABM, Section 4] where $Z_H^2(R, K)$ is defined to be the set of H -linear unital maps ν from $R \otimes R$ to K satisfying:

$$(\varepsilon_R \otimes \nu) * \nu(R \otimes m_R) = (\nu \otimes \varepsilon_R) * [\nu(m_R \otimes R)\Phi(\xi)],$$

and then $Z_H^2(R, K) = \Omega^2(Z_H^2(A, K))$; see [ABM, Theorem 4.10].

DEFINITION 3.1. Cohomology for $\text{Reg}(R^{\otimes t}, K)$ is defined in terms of ∂_A^t by:

$$(\partial_R^t)_+ := \Omega_{H,R}^{t+1} \circ (\partial_A^t)_+ \circ \mathcal{U}_{H,R}^t, \quad (\partial_R^t)_- := \Omega_{H,R}^{t+1} \circ (\partial_A^t)_- \circ \mathcal{U}_{H,R}^t, \quad \text{and} \quad \partial_R^t := \Omega_{H,R}^{t+1} \circ \partial_A^t \circ \mathcal{U}_{H,R}^t.$$

Thus the diagram below commutes as do similar diagrams with ∂ replaced by ∂_+ or ∂_- .

$$\begin{array}{ccccccc} \text{Reg}_{H,H}^b(K, K) & \xrightarrow{\partial_A^0} & \text{Reg}_{H,H}(A, K) & \xrightarrow{\partial_A^1} & \text{Reg}_{H,H}^b(A^{\otimes 2}, K) & \xrightarrow{\partial_A^2} & \text{Reg}_{H,H}^b(A^{\otimes 3}, K) \quad \dots \\ \Omega^0 \downarrow \uparrow \mathcal{U}^0 & & \Omega^1 \downarrow \uparrow \mathcal{U}^1 & & \Omega^2 \downarrow \uparrow \mathcal{U}^2 & & \Omega^3 \downarrow \uparrow \mathcal{U}^3 \\ \text{Reg}_H(K, K) & \xrightarrow{\partial_R^0} & \text{Reg}_H(R, K) & \xrightarrow{\partial_R^1} & \text{Reg}_H(R^{\otimes 2}, K) & \xrightarrow{\partial_R^2} & \text{Reg}_H(R^{\otimes 3}, K) \quad \dots \end{array}$$

LEMMA 3.2. For any $w \in \text{Reg}_H(R^{\otimes t}, K)$, $\partial_R^t(w) = (\partial_R^t)_+(w) * (\partial_R^t)_-(w^{-1})$.

Proof. We have

$$\begin{aligned} & (\partial_R^t)_+(w) * (\partial_R^t)_-(w^{-1}) = \left[\Omega_{H,R}^{t+1} (\partial_A^t)_+ \mathcal{U}_{H,R}^t(w) \right] * \left[\Omega_{H,R}^{t+1} (\partial_A^t)_- \mathcal{U}_{H,R}^t(w^{-1}) \right] \\ &= \Omega_{H,R}^{t+1} \left\{ \left[(\partial_A^t)_+ \mathcal{U}_{H,R}^t(w) \right] * \left[(\partial_A^t)_- ((\mathcal{U}_{H,R}^t(w))^{-1}) \right] \right\} = \Omega_{H,R}^{t+1} \partial_A^t \mathcal{U}_{H,R}^t(w) = \partial_R^t(w). \end{aligned}$$

□

In the following sections of this paper we will need formulas for $(\partial_R^2)_\pm$.

PROPOSITION 3.3. Let $w \in \text{Reg}_H(R^{\otimes t}, K)$ with $t = 0, 1, 2$.

i) For $t = 0$, we have

$$(\partial_R^0)_+(w) = w\varepsilon_R, \quad (\partial_R^0)_-(w) = w\varepsilon_R \quad \text{and} \quad \partial_R^0(w) = \varepsilon_R.$$

ii) For $t = 1$,

$$\begin{aligned} & (\partial_R^1)_+(w) = m_K(w \otimes w), \quad (\partial_R^1)_-(w) = wm_R \quad \text{and} \\ & \partial_R^1(w) = [m_K(w \otimes w)] * (w^{-1}m_R). \end{aligned}$$

iii) For $t = 2$,

$$\begin{aligned} & (\partial_R^2)_+(w) = [m_K(\varepsilon_R \otimes w)] * [w(R \otimes m_R)] = w(R \otimes m^w) * [m_K(\varepsilon_R \otimes w)], \\ & (\partial_R^2)_-(w) = [w(m_R \otimes R)\Phi(\xi)] * [m_K(w \otimes \varepsilon_R)], \\ & \partial_R^2(w) = [m_K(\varepsilon_R \otimes w)] * [w(R \otimes m_R)] * [w^{-1}(m_R \otimes R)\Phi(\xi)] * [m_K(w^{-1} \otimes \varepsilon_R)], \\ & \text{where (as in Lemma 2.10)} \quad m^w := m_R^w = w * m_R * w^{-1}. \end{aligned}$$

Proof. i) Since

$$(\partial_R^0)_\pm(w) = \Omega_{H,R}^1(\partial_A^0)_\pm \mathcal{U}_{H,R}^0(w) = \Omega_{H,R}^1(\partial_A^0)_\pm(w) = \Omega_{H,R}^1(w\varepsilon_A) = w\varepsilon_R,$$

we obtain $\partial_R^0(w) := (\partial_R^0)_+(w) * (\partial_R^0)_-(w^{-1}) = w\varepsilon_R * w^{-1}\varepsilon_R = \varepsilon_R$.

ii) We compute

$$\begin{aligned} & (\partial_R^1)_+(w) = \Omega_{H,R}^2(\partial_A^1)_+ \mathcal{U}_{H,R}^1(w) = \Omega_{H,R}^2(\partial_A^1)_+(w \otimes \varepsilon_H) \\ &= \Omega_{H,R}^2[m_K(w \otimes \varepsilon_H \otimes w \otimes \varepsilon_H)] = m_K(w \otimes w), \quad \text{and} \\ & (\partial_R^1)_-(w) = \Omega_{H,R}^2(\partial_A^1)_- \mathcal{U}_{H,R}^1(w) = \Omega_{H,R}^2(\partial_A^1)_-(w \otimes \varepsilon_H) \\ &= \Omega_{H,R}^2((w \otimes \varepsilon_H)m_A) = (w \otimes \varepsilon_H)m_A(R \otimes u_H \otimes R \otimes u_H) \stackrel{(21)}{=} wm_R \end{aligned}$$

so that $\partial_R^1(w) = (\partial_R^1)_+(w) * (\partial_R^1)_-(w^{-1}) = m_K(w \otimes w) * w^{-1}m_R$.

iii) We first show that

$$(23) \quad m_K \circ (\varepsilon_A \otimes \mathcal{U}_{H,R}^2(w)) = \mathcal{U}_{H,R}^3[m_K \circ (\varepsilon_R \otimes w)],$$

$$(24) \quad \mathcal{U}_{H,R}^2(w) \circ (A \otimes m_A) = \mathcal{U}_{H,R}^3[w \circ (R \otimes m_R)].$$

By the bijectivity of $\Omega_{H,R}^3$, showing that (23) holds is equivalent to showing that $\Omega_{H,R}^3(\text{lhs}(23)) = \Omega_{H,R}^3(\text{rhs}(23))$. Then it is only necessary to check (23) on elements of the form $(r\#1_H) \otimes (s\#1_H) \otimes (t\#1_H)$ with $r, s, t \in R$:

$$\begin{aligned} & m_K(\varepsilon_A \otimes \mathcal{U}_{H,R}^2(w))[(r\#1_H) \otimes (s\#1_H) \otimes (t\#1_H)] \\ &= \varepsilon_A(r\#1_H) \mathcal{U}_{H,R}^2(w)[(s\#1_H) \otimes (t\#1_H)] = \varepsilon_R(r) w(s \otimes t) = m_K(\varepsilon_R \otimes w)(r \otimes s \otimes t) \\ &= \mathcal{U}_{H,R}^3[m_K(\varepsilon_R \otimes w)][(r\#1_H) \otimes (s\#1_H) \otimes (t\#1_H)]. \end{aligned}$$

Similarly, to prove (24), we compute:

$$\begin{aligned} & \mathcal{U}_{H,R}^2(w)(A \otimes m_A)[(r\#1_H) \otimes (s\#1_H) \otimes (t\#1_H)] \\ &= \mathcal{U}_{H,R}^2(w)[(r\#1_H) \otimes m_A[(s\#1_H) \otimes (t\#1_H)]] \\ &= w[r \otimes (R \otimes \varepsilon_H) m_A[(s\#1_H) \otimes (t\#1_H)]] \stackrel{(21)}{=} w[r \otimes m_R(s \otimes t)] \\ &= w(R \otimes m_R)(r \otimes s \otimes t) = \mathcal{U}_{H,R}^3[w(R \otimes m_R)][(r\#1_H) \otimes (s\#1_H) \otimes (t\#1_H)]. \end{aligned}$$

Since $\mathcal{U}_{H,R}^3$ is a convolution preserving isomorphism, using the definitions of $(\partial_R^2)_+$ and $(\partial_A^2)_+$,

$$\begin{aligned} \left[\mathcal{U}_{H,R}^3 \circ (\partial_R^2)_+ \right](w) &\stackrel{\text{def}}{=} \left[(\partial_A^2)_+ \circ \mathcal{U}_{H,R}^2 \right](w) \\ &\stackrel{\text{def}}{=} \left[m_K(\varepsilon_A \otimes \mathcal{U}_{H,R}^2(w)) * \mathcal{U}_{H,R}^2(w)(A \otimes m_A) \right] \\ &\stackrel{(23),(24)}{=} \mathcal{U}_{H,R}^3[m_K(\varepsilon_R \otimes w)] * \mathcal{U}_{H,R}^3[w(R \otimes m_R)] \\ &= \mathcal{U}_{H,R}^3[m_K(\varepsilon_R \otimes w) * w(R \otimes m_R)] \end{aligned}$$

so that $(\partial_R^2)_+(w) = [m_K(\varepsilon_R \otimes w)] * [w(R \otimes m_R)]$ as claimed. It is straightforward to verify the second equality for $(\partial_R^2)_+(w)$, or one can use (8).

Similarly, to find the formula for $(\partial_R^2)_-$, we first prove

$$(25) \quad [\mathcal{U}_{H,R}^2(w)] \circ (m_A \otimes A) = \mathcal{U}_{H,R}^3[w \circ (m_R \otimes R) \circ \Phi(\xi)],$$

$$(26) \quad m_K \circ (\mathcal{U}_{H,R}^2(w) \otimes \varepsilon_A) = \mathcal{U}_{H,R}^3[m_K \circ (w \otimes \varepsilon_R)].$$

For $r, s, t \in R$

$$\begin{aligned} & \mathcal{U}_{H,R}^2(w)(m_A \otimes A)[(r\#1_H) \otimes (s\#1_H) \otimes (t\#1_H)] \\ &= \mathcal{U}_{H,R}^2(w)[m_A[(r\#1_H) \otimes (s\#1_H)] \otimes (t\#1_H)] \\ &= \mathcal{U}_{H,R}^2(w) \left[m_R \left(r^{(1)} \otimes r_{(-1)}^{(2)} s^{(1)} \right) \# \xi \left(r_{(0)}^{(2)} \otimes s^{(2)} \right) \otimes (t\#1_H) \right] \\ &= w \left[m_R \left(r^{(1)} \otimes r_{(-1)}^{(2)} s^{(1)} \right) \otimes \xi \left(r_{(0)}^{(2)} \otimes s^{(2)} \right) t \right] \\ &= w(m_R \otimes R)(R \otimes R \otimes \mu_R)(R \otimes R \otimes \xi \otimes R)(\Delta_{R \otimes R} \otimes R)[r \otimes s \otimes t] \\ &= w(m_R \otimes R) \Phi(\xi)[r \otimes s \otimes t] \\ &= \mathcal{U}_{H,R}^3[w(m_R \otimes R) \Phi(\xi)][(r\#1_H) \otimes (s\#1_H) \otimes (t\#1_H)], \end{aligned}$$

and

$$\begin{aligned} & m_K(\mathcal{U}_{H,R}^2(w) \otimes \varepsilon_A)[(r\#1_H) \otimes (s\#1_H) \otimes (t\#1_H)] \\ &= \mathcal{U}_{H,R}^2(w)[(r\#1_H) \otimes (s\#1_H)] \varepsilon_A(t\#1_H) \\ &= w(r \otimes s) \varepsilon_R(t) \\ &= m_K(w \otimes \varepsilon_R)(r \otimes s \otimes t) \end{aligned}$$

$$= \mathcal{U}_{H,R}^3 [m_K (w \otimes \varepsilon_R)] [(r\#1_H) \otimes (s\#1_H) \otimes (t\#1_H)]$$

so that (25) and (26) hold. Since $\mathcal{U}_{H,R}^3$ is a convolution preserving isomorphism, using the definitions of $(\partial_R^2)_+$ and $(\partial_A^2)_+$, we get

$$\begin{aligned} & \left[\mathcal{U}_{H,R}^3 \circ (\partial_R^2)_- \right] (w) \stackrel{\text{def}}{=} \left[(\partial_A^2)_- \circ \mathcal{U}_{H,R}^2 \right] (w) \\ & \stackrel{\text{def}}{=} \mathcal{U}_{H,R}^2 (w) (m_A \otimes A) * m_K (\mathcal{U}_{H,R}^2 (w) \otimes \varepsilon_A) \\ & \stackrel{(25),(26)}{=} \mathcal{U}_{H,R}^3 [w (m_R \otimes R) \Phi(\xi)] * \mathcal{U}_{H,R}^3 [m_K (w \otimes \varepsilon_R)] \\ & = \mathcal{U}_{H,R}^3 [w (m_R \otimes R) \Phi(\xi) * m_K (w \otimes \varepsilon_R)] \end{aligned}$$

so that the formula for $(\partial_R^2)_-(w)$ is proved and the formula for $\partial_R^2(w)$ follows immediately. \square

COROLLARY 3.4. *Suppose that $w \in \text{Reg}_H(R \otimes R, K)$ is such that $\xi = u_H w^{-1} * \Psi(w)$. Then for $m^w = w * m_R * w^{-1}$ as in the proposition, $(\partial_R^2)_-(w^{-1}) = (w^{-1} \otimes \varepsilon_R) * w^{-1}(m^w \otimes R)$ and thus*

$$(\partial_R^2)(w) = w(R \otimes m^w) * (\varepsilon_R \otimes w) * (w^{-1} \otimes \varepsilon_R) * w^{-1}(m^w \otimes R).$$

Proof. By Proposition 3.3, since $\xi = u_H w^{-1} * \Psi(w)$, we have:

$$\begin{aligned} (\partial_R^2)_-(w^{-1}) &= [w^{-1} (m_R \otimes R) \Phi(\xi)] * (w^{-1} \otimes \varepsilon_R) \\ &= \left[w^{-1} \left((m^w)^{w^{-1}} \otimes R \right) \Phi(u_H w^{-1} * \Psi(w)) \right] * (w^{-1} \otimes \varepsilon_R) \\ &\stackrel{(8)}{=} [(w^{-1} \otimes \varepsilon_R) * w^{-1}(m^w \otimes R) * (w \otimes \varepsilon_R)] * (w^{-1} \otimes \varepsilon_R) \\ &= (w^{-1} \otimes \varepsilon_R) * w^{-1}(m^w \otimes R). \end{aligned}$$

Since by Proposition 3.3, $(\partial_R^2)_+(w) = w(R \otimes m^w) * (\varepsilon_R \otimes w)$, the result follows. \square

4. A NEW INTERPRETATION FOR THE COCYCLE ξ FOR A PRE-BIALGEBRA R

Throughout this section $(R, m, u, \Delta, \varepsilon)$ will denote a connected pre-bialgebra in ${}^H_H\mathcal{YD}$. Define

$$\Xi := \{\xi | \xi \text{ is a cocycle for } R\} = \{\xi | \xi \text{ satisfies (12) – (17) in Section 2.3}\},$$

in other words, the set of ξ such that (R, ξ) is a pre-bialgebra with cocycle so that a bosonization $A := R\#_\xi H$ can be built. By Remark 2.7, and by (12), $\Xi \subset \text{Reg}_H(R \otimes R, H)$ where H has the left adjoint action. In this section, we establish a bijective correspondence between Ξ and a set $V \subset \text{Reg}_H(R \otimes R, K)$.

First for C any connected coaugmented coalgebra in ${}^H_H\mathcal{YD}$, we find a set in $\text{Reg}_H(C, K)$ in bijective correspondence with the set of normalized dual Sweedler 1-cocycles in $\text{Reg}_H(C, H)$ which map 1_C to 1_H . Recall that λ denotes the ad-invariant integral for H , recall from Definition 2.3 that $\Psi(v) = (H \otimes v) \circ \rho_C$, for v in $\text{Hom}(C, K)$ and define the following subsets of $\text{Reg}_H(C, K)$ and $\text{Reg}_H(C, H)$:

$$\mathcal{G} = \{v \in \text{Hom}_H(C, K) \mid v(1_C) = 1_K \text{ and } \lambda \circ [\Psi(v)] = \lambda \circ (H \otimes v) \circ \rho_C = \varepsilon_C\}, \text{ and}$$

$$\begin{aligned} \mathcal{S} &= \{\xi \in \text{Hom}_H(C, H) \mid \xi \text{ is a normalized dual Sweedler 1-cocycle, and } \xi(1_C) = 1_H\} \\ &= \{\xi \in \text{Hom}_H(C, H) \mid \Delta_H \xi = (m_H \otimes \xi)(\xi \otimes \rho_C) \Delta_C, \varepsilon_H \xi = \varepsilon_C \text{ and } \xi u_C := u_H\}. \end{aligned}$$

By Remarks 2.7 and 2.9 the sets \mathcal{G} and \mathcal{S} consist of convolution invertible maps.

THEOREM 4.1. *For $C, \mathcal{G}, \mathcal{S}$ as above, the maps $F : \mathcal{G} \rightarrow \mathcal{S}$ and $G : \mathcal{S} \rightarrow \mathcal{G}$ defined by*

$$F(v) = u_H v^{-1} * \Psi(v) \quad \text{and} \quad G(\xi) = (\lambda \xi)^{-1}$$

are inverse bijections.

Proof. Let $v \in \mathcal{G}$, and $\xi := u_H v^{-1} * \Psi(v)$. We check that $\xi \in \mathcal{S}$. First we see that $\xi u_C = u_H$ since

$$\xi(1_C) = [u_H v^{-1} * \Psi(v)](1_C) = u_H v^{-1}(1_C) \Psi(v)(1_C) = 1_H.$$

Also

$$\varepsilon_H \xi = \varepsilon_H [u_H v^{-1} * \Psi(v)] = \varepsilon_H u_H v^{-1} * \varepsilon_H \Psi(v) = v^{-1} * v = \varepsilon_C.$$

Moreover, for $x \in C$

$$\begin{aligned} (m_H \otimes \xi)(\xi \otimes \rho_C) \Delta_C(x) &= (m_H \otimes \xi)(\xi(x^{(1)}) \otimes x_{\langle -1 \rangle}^{(2)} \otimes x_{\langle 0 \rangle}^{(2)}) \\ &= \xi(x^{(1)}) x_{\langle -1 \rangle}^{(2)} \otimes \xi(x_{\langle 0 \rangle}^{(2)}) = [[u_H v^{-1} * \Psi(v)](x^{(1)})] x_{\langle -1 \rangle}^{(2)} \otimes [u_H v^{-1} * \Psi(v)](x_{\langle 0 \rangle}^{(2)}) \\ &= u_H v^{-1}(x^{(1)}) \Psi(v)(x_{\langle 0 \rangle}^{(2)}) x_{\langle -1 \rangle}^{(3)} \otimes u_H v^{-1}[(x_{\langle 0 \rangle}^{(3)})^{(1)}] \Psi(v)[(x_{\langle 0 \rangle}^{(3)})^{(2)}] \\ &\stackrel{(2)}{=} u_H v^{-1}(x^{(1)}) \Psi(v)(x_{\langle 0 \rangle}^{(2)}) x_{\langle -1 \rangle}^{(3)} x_{\langle -1 \rangle}^{(4)} \otimes u_H v^{-1}(x_{\langle 0 \rangle}^{(3)}) \Psi(v)(x_{\langle 0 \rangle}^{(4)}) \\ &= v^{-1}(x^{(1)}) x_{\langle -1 \rangle}^{(2)} v(x_{\langle 0 \rangle}^{(2)}) x_{\langle -1 \rangle}^{(3)} x_{\langle -1 \rangle}^{(4)} \otimes v^{-1}(x_{\langle 0 \rangle}^{(3)}) \Psi(v)(x_{\langle 0 \rangle}^{(4)}) \\ &= v^{-1}(x^{(1)}) x_{\langle -1 \rangle}^{(2)} x_{\langle -1 \rangle}^{(3)} x_{\langle -1 \rangle}^{(4)} \otimes v(x_{\langle 0 \rangle}^{(2)}) v^{-1}(x_{\langle 0 \rangle}^{(3)}) \Psi(v)(x_{\langle 0 \rangle}^{(4)}) \\ &\stackrel{(2)}{=} v^{-1}(x^{(1)}) x_{\langle -1 \rangle}^{(2)} \otimes v((x_{\langle 0 \rangle}^{(2)})^{(1)}) v^{-1}((x_{\langle 0 \rangle}^{(2)})^{(2)}) \Psi(v)((x_{\langle 0 \rangle}^{(2)})^{(3)}) \\ &= v^{-1}(x^{(1)}) x_{\langle -1 \rangle}^{(2)} \otimes (v * v^{-1})((x_{\langle 0 \rangle}^{(2)})^{(1)}) \Psi(v)((x_{\langle 0 \rangle}^{(2)})^{(2)}) \\ &= v^{-1}(x^{(1)}) x_{\langle -1 \rangle}^{(2)} \otimes \Psi(v)(x_{\langle 0 \rangle}^{(2)}) = v^{-1}(x^{(1)}) x_{\langle -2 \rangle}^{(2)} \otimes x_{\langle -1 \rangle}^{(2)} v(x_{\langle 0 \rangle}^{(2)}) \\ &= \Delta_H[v^{-1}(x^{(1)}) x_{\langle -1 \rangle}^{(2)} v(x_{\langle 0 \rangle}^{(2)})] = \Delta_H \xi(x). \end{aligned}$$

Thus F maps \mathcal{G} to \mathcal{S} . Now let $\xi \in \mathcal{S}$. Then, $\lambda \xi u_C = \text{Id}_K$ so that, by Remark 2.9, $\lambda \xi$ is convolution invertible and $(\lambda \xi)^{-1} u_C = \text{Id}_K$. Thus it makes sense to consider $v := (\lambda \xi)^{-1}$ and to see that $v \in \mathcal{G}$, it remains to show that $\lambda \circ \Psi(v) = \varepsilon_C$. Since λ is a left integral, $u_H \lambda = (H \otimes \lambda) \Delta_H$. Thus,

$$\begin{aligned} u_H v^{-1} &= u_H \lambda \xi = (H \otimes \lambda) \Delta_H \xi \stackrel{(7)}{=} (m_H \otimes \lambda \xi)(\xi \otimes \rho_C) \Delta_C \\ &= (m_H \otimes v^{-1})(\xi \otimes \rho_C) \Delta_C = \xi * \Psi(v^{-1}) \end{aligned}$$

so that $\xi = u_H v^{-1} * \Psi(v)$. Hence we have $\lambda \Psi(v) = \lambda[u_H v * \xi] = v * \lambda \xi = \varepsilon_C$. Thus $v \in \mathcal{G}$ and $\xi = F(v)$. This proves that G maps \mathcal{S} to \mathcal{G} and $F \circ G = \text{Id}_{\mathcal{S}}$.

Next we show that $G \circ F = \text{Id}_{\mathcal{G}}$. Let $v \in \mathcal{G}$. Let $\xi := u_H v^{-1} * \Psi(v) = F(v) \in \mathcal{S}$. We check that $v = (\lambda \xi)^{-1}$. We have

$$\lambda \xi = \lambda \circ [u_H v^{-1} * \Psi(v)] = v^{-1} * \lambda \Psi(v) = v^{-1} * \varepsilon_C = v^{-1}.$$

Thus $v = (\lambda \xi)^{-1} = (G \circ F)(v)$.

Finally suppose that $v \in \mathcal{G}$ is left H -linear, i.e., $v(h \cdot c) = \varepsilon_H(h) v(c)$. Then v^{-1} and $\Psi(v)$ are also left H -linear by Remark 2.9, and Lemma 2.5 so that their convolution product $F(v)$ is left H -linear, as required. Conversely, if ξ is left H -linear, so is $\lambda \xi$ and thus so is $G(\xi)$. \square

We note that if C is coaugmented but not necessarily connected, then F is still a map from $\text{Reg}_H(C, K)$ to the set \mathcal{S} .

Recall that if (R, ξ) is a pre-bialgebra with cocycle, the cohomology of (R, ξ) is defined using the cohomology of $R \#_{\xi} H$. By Corollary 3.4, if $v \in \text{Reg}_H(R \otimes R, K)$ is such that $\xi = u_H v^{-1} * \Psi(v) = F(v)$ then $\partial_R^2(v) = v(R \otimes m^v) * (\varepsilon_R \otimes v) * (v^{-1} \otimes \varepsilon_R) * v^{-1}(m^v \otimes R)$. However, this expression makes sense for $v \in \text{Reg}_H(R \otimes R, K)$ even if $u_H v^{-1} * \Psi(v) \notin \Xi$.

DEFINITION 4.2. Let $(R, m, u, \Delta, \varepsilon)$ be a connected pre-bialgebra in ${}^H_H \mathcal{YD}$. For $w \in \text{Reg}_H(R \otimes R, K)$ define $\alpha(w) = \alpha_+(w) * \alpha_-(w^{-1})$ where:

$$\alpha_+(w) := w(R \otimes m^w) * (\varepsilon \otimes w), \text{ and } \alpha_-(w^{-1}) = (w^{-1} \otimes \varepsilon) * w^{-1}(m^w \otimes R).$$

Theorem 4.1 establishes a bijection between $\mathcal{S} \supset \Xi$ and a set $\mathcal{G} \subset \text{Reg}_H(R \otimes R, K)$. The rest of this section will be devoted to proving the following theorem:

THEOREM 4.3. *Let $(R, m, u, \Delta, \varepsilon)$ be a connected pre-bialgebra in ${}^H_H\mathcal{YD}$, and let $\mathcal{S}, \mathcal{G}, F, G$ be as in Theorem 4.1. Then $G(\Xi) = V$ where $V \subset \mathcal{G}$ is the set of v in \mathcal{G} satisfying the following:*

$$(27) \quad m^v \text{ is } H\text{-colinear, i.e. } \rho_R m^v = (H \otimes m^v) \rho_{R \otimes R},$$

$$(28) \quad m^v(R \otimes m^v) * \alpha(v) = \alpha(v) * m^v(m^v \otimes R),$$

$$(29) \quad \Psi(\alpha(v)) = u_H \alpha(v),$$

$$(30) \quad v \text{ is unital, i.e. for all } r \in R, v(r \otimes 1_R) = v(1_R \otimes r) = \varepsilon(r).$$

Note that (28) says that m^v is associative up to inner action by the invertible element $\alpha(v)$.

REMARK 4.4. We note that condition (29) means that $\alpha(v) : R^{\otimes 3} \rightarrow K$ is left H -colinear. Since $\alpha(v)$ is also left H -linear, $\alpha(v)$ is in ${}^H_H\mathcal{YD}$. If in addition v is unital, then $\alpha(v)$ is unital so that one gets that $\alpha(v)$ is a retraction of the unit $u_{R^{\otimes 3}} : K \rightarrow R^{\otimes 3}$ of the coaugmented coalgebra $R^{\otimes 3}$ in the category ${}^H_H\mathcal{YD}$.

The proof of Theorem 4.3 consists of a series of propositions equating the conditions (14) to (17) from Definition 2.12 to the conditions (27) to (30) listed above.

By Lemma 2.13, condition (14) is equivalent to (18), namely that for all $z \in C := R \otimes R$,

$$\rho_R(m(z)) = m(z)_{\langle -1 \rangle} \otimes m(z)_{\langle 0 \rangle} = \xi(z^{(1)}) z_{\langle -1 \rangle}^{(2)} \xi^{-1}(z^{(3)}) \otimes m(z_{\langle 0 \rangle}^{(2)}).$$

For $v \in \text{Reg}_H(R \otimes R, K)$, m^v is H -linear, being the product of left H -linear maps. The next proposition shows that (14) (equivalently (18)) holds for $\xi \in \mathcal{S}$ if and only if m^v is left H -colinear where $v = G(\xi)$.

PROPOSITION 4.5. *Let $v \in \mathcal{G}$ and let $\xi := F(v) = u_H v^{-1} * \Psi(v)$. Then ξ satisfies (14) if and only if m^v is left H -colinear, i.e.,*

$$\rho_R m^v = (H \otimes m^v) \rho_{R \otimes R}.$$

Proof. Since $\xi = u_H v^{-1} * \Psi(v)$ then $\xi^{-1} = \Psi(v^{-1}) * u_H v$, where $\Psi(\gamma) = (H \otimes \gamma) \rho_C$ and $C = R \otimes R$. Hence $\xi^{-1}(z) = z_{\langle -1 \rangle}^{(1)} v^{-1}(z_{\langle 0 \rangle}^{(1)}) v(z^{(2)})$. Then the right side of (18) applied to $z \in C$ is:

$$\begin{aligned} & \xi(z^{(1)}) z_{\langle -1 \rangle}^{(2)} \xi^{-1}(z^{(3)}) \otimes m(z_{\langle 0 \rangle}^{(2)}) \\ &= [v^{-1}(z^{(1)}) z_{\langle -1 \rangle}^{(2)} v(z_{\langle 0 \rangle}^{(2)})] z_{\langle -1 \rangle}^{(3)} [z_{\langle -1 \rangle}^{(4)} v^{-1}(z_{\langle 0 \rangle}^{(4)}) v(z^{(5)})] \otimes m(z_{\langle 0 \rangle}^{(3)}) \\ &= v^{-1}(z^{(1)}) z_{\langle -1 \rangle}^{(2)} z_{\langle -1 \rangle}^{(3)} z_{\langle -1 \rangle}^{(4)} v(z^{(5)}) \otimes v(z_{\langle 0 \rangle}^{(2)}) m(z_{\langle 0 \rangle}^{(3)}) v^{-1}(z_{\langle 0 \rangle}^{(4)}) \\ &\stackrel{(2)}{=} v^{-1}(z^{(1)}) z_{\langle -1 \rangle}^{(2)} v(z^{(3)}) \otimes v((z_{\langle 0 \rangle}^{(2)})^{(1)}) m((z_{\langle 0 \rangle}^{(2)})^{(2)}) v^{-1}((z_{\langle 0 \rangle}^{(2)})^{(3)}) \\ &= v^{-1}(z^{(1)}) z_{\langle -1 \rangle}^{(2)} v(z^{(3)}) \otimes (v * m * v^{-1})(z_{\langle 0 \rangle}^{(2)}) \\ &= z_{\langle -1 \rangle}^{(2)} \otimes m^v(v^{-1}(z^{(1)}) z_{\langle 0 \rangle}^{(2)} v(z^{(3)})) \\ &= (H \otimes m^v) \rho_C(v^{-1}(z^{(1)}) z_{\langle 0 \rangle}^{(2)} v(z^{(3)})) \\ &= (H \otimes m^v) \rho_C(u_C v^{-1} * \text{Id}_C * u_C v)(z). \end{aligned}$$

so that

$$(31) \quad \xi(z^{(1)}) z_{\langle -1 \rangle}^{(2)} \xi^{-1}(z^{(3)}) \otimes m(z_{\langle 0 \rangle}^{(2)}) = (H \otimes m^v) \rho_C(u_C v^{-1} * \text{Id}_C * u_C v)(z).$$

Note that $(u_C v^{-1} * \text{Id}_C * u_C v)$ and $(u_C v * \text{Id}_C * u_C v^{-1})$ are composition inverses. Thus if (18) holds, by (31) we have

$$\rho_R m = (H \otimes m^v) \rho_C(u_C v^{-1} * \text{Id}_C * u_C v)$$

whence

$$\rho_R m^v = \rho_R m(u_C v * \text{Id}_C * u_C v^{-1}) = (H \otimes m^v) \rho_C$$

as required. Conversely, if (27) holds, then

$$\xi(z^{(1)}) z_{\langle -1 \rangle}^{(2)} \xi^{-1}(z^{(3)}) \otimes m(z_{\langle 0 \rangle}^{(2)}) \stackrel{(31)}{=} \rho_R m^v(u_C v^{-1} * \text{Id}_C * u_C v)(z) = \rho_R m(z).$$

□

Now we can see the relationship between the associativity condition (15) on m_R with $\xi = F(v)$ and the associativity of the multiplication m^v on R .

PROPOSITION 4.6. *Let $v \in \mathcal{G}$ and assume that v satisfies (27) so that m^v is left H -colinear. Let $\xi := F(v) = v^{-1} * \Psi(v)$. Then (15) holds for ξ if and only if (28) holds for v , i.e.,*

$$m^v(R \otimes m^v) * \alpha(v) = \alpha(v) * m^v(m^v \otimes R),$$

where $\alpha(v)$ was defined in Definition 4.2.

Proof. Since m^v is left H -linear and colinear, we may apply (8) and (6). We have

$$\begin{aligned} m(R \otimes m) &= m(R \otimes (v^{-1} * m^v * v)) \\ &\stackrel{(8)}{=} (\varepsilon \otimes v^{-1}) * m(R \otimes m^v) * (\varepsilon \otimes v) \\ &= (\varepsilon \otimes v^{-1}) * (v^{-1} * m^v * v)(R \otimes m^v) * (\varepsilon \otimes v) \\ &\stackrel{(6)}{=} (\varepsilon \otimes v^{-1}) * v^{-1}(R \otimes m^v) * m^v(R \otimes m^v) * v(R \otimes m^v) * (\varepsilon \otimes v) \end{aligned}$$

On the other hand,

$$\begin{aligned} m(m \otimes R)\Phi(\xi) &= m(m \otimes R)\Phi(v^{-1} * \Psi(v)) \\ &\stackrel{(8)}{=} (v^{-1} \otimes \varepsilon) * m(m^v \otimes R) * (v \otimes \varepsilon) \\ &= (v^{-1} \otimes \varepsilon) * (v^{-1} * m^v * v)(m^v \otimes R) * (v \otimes \varepsilon) \\ &\stackrel{(6)}{=} (v^{-1} \otimes \varepsilon) * v^{-1}(m^v \otimes R) * m^v(m^v \otimes R) * v(m^v \otimes R) * (v \otimes \varepsilon). \end{aligned}$$

Since for any coalgebra map $\varphi : R \otimes R \otimes R \rightarrow R \otimes R$, $v \circ \varphi$ and $v^{-1} \circ \varphi$ are convolution inverses, the statement follows. \square

PROPOSITION 4.7. *Let $v \in \mathcal{G}$ and assume that v satisfies (27) so that m^v is left H -colinear. Set $\xi := F(v) = v^{-1} * \Psi(v)$. Then (16) holds for ξ if and only if (29) holds for v , i.e.,*

$$\Psi(\alpha(v)) = u_H \alpha(v).$$

Proof. First consider the left side of (16). For $r \in R$ and $w \in R \otimes R$, we have

$$\begin{aligned} &m_H(\xi \otimes H)[R \otimes (m \otimes \xi) \Delta_{R \otimes R}](r \otimes w) \\ &= \xi \left[r \otimes m(w^{(1)}) \right] \xi(w^{(2)}) \\ &= \xi \left[r \otimes m(w^{(1)}) \right] [u_H v^{-1} * \Psi(v)](w^{(2)}) \\ &= \xi \left[r \otimes m(w^{(1)}) \right] v^{-1}(w^{(2)}) \Psi(v)(w^{(3)}) \\ &= \xi \left[r \otimes (m * v^{-1})(w^{(1)}) \right] \Psi(v)(w^{(2)}) \\ &= \xi \left[r \otimes (v^{-1} * m^v)(w^{(1)}) \right] \Psi(v)(w^{(2)}) \\ &= v^{-1}(w^{(1)}) \xi \left[r \otimes m^v(w^{(2)}) \right] \Psi(v)(w^{(3)}) \\ &= v^{-1}(w^{(1)}) \xi(R \otimes m^v)(r \otimes w^{(2)}) \Psi(v)(w^{(3)}) \\ &= v^{-1}(w^{(1)}) [u_H v^{-1} * \Psi(v)](R \otimes m^v)(r \otimes w^{(2)}) \Psi(v)(w^{(3)}) \\ &\stackrel{(6)}{=} v^{-1}(w^{(1)}) [u_H v^{-1}(R \otimes m^v) * \Psi(v)(R \otimes m^v)](r \otimes w^{(2)}) \Psi(v)(w^{(3)}) \\ &= (\varepsilon \otimes v^{-1})(r^{(1)} \otimes r_{(-1)}^{(2)} w^{(1)}) [u_H v^{-1}(R \otimes m^v) * \Psi(v)(R \otimes m^v)](r_{(0)}^{(2)} \otimes w^{(2)}) \Psi(v)(w^{(3)}) \\ &= [u_H(\varepsilon \otimes v^{-1}) * u_H v^{-1}(R \otimes m^v) * \Psi(v)(R \otimes m^v)](r \otimes w^{(1)}) \Psi(v)(w^{(2)}) \\ &= [u_H(\alpha)_+(v)^{-1} * \Psi(v)(R \otimes m^v)](r^{(1)} \otimes r_{(-1)}^{(2)} w^{(1)}) \Psi(\varepsilon \otimes v)(r_{(0)}^{(2)} \otimes w^{(2)}) \end{aligned}$$

$$\begin{aligned}
& \stackrel{(2.4(i))}{=} \left[u_H(\alpha)_+(v)^{-1} * \Psi(v(R \otimes m^v)) * \Psi(\varepsilon \otimes v) \right] (r \otimes w) \\
& \stackrel{(2.4(ii))}{=} \left[u_H(\alpha)_+(v)^{-1} * \Psi[v(R \otimes m^v) * (\varepsilon \otimes v)] \right] (r \otimes w) \\
& = \left[u_H(\alpha)_+(v)^{-1} * \Psi((\alpha)_+(v)) \right] (r \otimes w).
\end{aligned}$$

so that the left hand side of (16) equals $u_H(\alpha)_+(v)^{-1} * \Psi((\alpha)_+(v))$. Now consider the right side of (16). For $z \in R \otimes R$ and $t \in R$, we have

$$\begin{aligned}
& m_H(\xi \otimes H)(R \otimes c_{H,R})[(m \otimes \xi) \Delta_{R \otimes R} \otimes R](z \otimes t) \\
& = \xi \left[m(z^{(1)}) \otimes \xi(z^{(2)})_{(1)} t \right] \xi(z^{(2)})_{(2)} \\
& = \xi \left[m(z^{(1)}) \otimes \left\{ [u_H v^{-1} * \Psi(v)](z^{(2)}) \right\}_{(1)} t \right] \left\{ [u_H v^{-1} * \Psi(v)](z^{(2)}) \right\}_{(2)} \\
& = \xi \left[m(z^{(1)}) \otimes v^{-1}(z^{(2)}) z_{(-2)}^{(3)} t \right] z_{(-1)}^{(3)} v(z_{(0)}^{(3)}) \\
& = \xi \left[(m_R * v^{-1})(z^{(1)}) \otimes z_{(-1)}^{(2)} t \right] \Psi(v)(z_{(0)}^{(2)}) \\
& = \xi \left[(v^{-1} * m^v)(z^{(1)}) \otimes z_{(-1)}^{(2)} t \right] \Psi(v)(z_{(0)}^{(2)}) \\
& = v^{-1}(z^{(1)}) \xi \left[m^v(z^{(2)}) \otimes z_{(-1)}^{(3)} t \right] \Psi(v)(z_{(0)}^{(3)}) \\
& = v^{-1}(z^{(1)}) \xi(m^v \otimes R)(z^{(2)} \otimes z_{(-1)}^{(3)} t) \Psi(v)(z_{(0)}^{(3)}) \\
& = v^{-1}(z^{(1)}) [u_H v^{-1} * \Psi(v)](m^v \otimes R)(z^{(2)} \otimes z_{(-1)}^{(3)} t) \Psi(v)(z_{(0)}^{(3)}) \\
& \stackrel{(6)}{=} v^{-1}(z^{(1)}) [u_H v^{-1}(m^v \otimes R) * \Psi(v)(m^v \otimes R)](z^{(2)} \otimes z_{(-1)}^{(3)} t) \Psi(v)(z_{(0)}^{(3)}) \\
& \stackrel{(2.4(i))}{=} v^{-1}(z^{(1)}) [u_H v^{-1}(m^v \otimes R) * \Psi(v(m^v \otimes R))](z^{(2)} \otimes z_{(-1)}^{(3)} t^{(1)}) \Psi(v \otimes \varepsilon)(z_{(0)}^{(3)} \otimes t^{(2)}) \\
& = v^{-1}(z^{(1)}) [u_H v^{-1}(m^v \otimes R) * \Psi(v(m^v \otimes R)) * \Psi(v \otimes \varepsilon)](z^{(2)} \otimes t) \\
& \stackrel{(6)}{=} v^{-1}(z^{(1)}) [u_H v^{-1}(m^v \otimes R) * \Psi((\alpha)_-(v^{-1})^{-1})](z^{(2)} \otimes t) \\
& = (v^{-1} \otimes \varepsilon)(z^{(1)} \otimes z_{(-1)}^{(2)} t^{(1)}) [u_H v^{-1}(m^v \otimes R) * \Psi((\alpha)_-(v^{-1})^{-1})](z_{(0)}^{(2)} \otimes t^{(2)}) \\
& = [u_H(v^{-1} \otimes \varepsilon) * u_H v^{-1}(m^v \otimes R) * \Psi((\alpha)_-(v^{-1})^{-1})](z \otimes t) \\
& = [u_H(\alpha)_-(v^{-1}) * \Psi((\alpha)_-(v^{-1})^{-1})](z \otimes t)
\end{aligned}$$

so that the right hand side of (16) is $u_H(\alpha)_-(v^{-1}) * \Psi((\alpha)_-(v^{-1})^{-1})$. By Remark 2.4(ii), Ψ is an algebra map, and the statement follows. \square

Finally we show that ξ is unital if and only if $v = G(\xi)$ is unital.

PROPOSITION 4.8. *Let $v \in \mathcal{G}$ and let $\xi : F(v) = u_H v^{-1} * \Psi(v)$. Then ξ satisfies (17) if and only if v is unital, i.e., for all $r \in R$, $v(1_R \otimes r) = v(r \otimes 1_R) = \varepsilon(r)$.*

Proof. Note that v is unital if and only if v^{-1} is. Since for all $r \in R$,

$$\xi(r \otimes 1_R) = [u_H v^{-1} * \Psi(v)](r \otimes 1_R) = v^{-1}(r^{(1)} \otimes 1_R) r_{(-1)}^{(2)} v(r_{(0)}^{(2)} \otimes 1_R),$$

then v unital, i.e. (30), implies $\xi(r \otimes 1_R) = \varepsilon(r) 1_H$.

Conversely, if $\xi(r \otimes 1) = \varepsilon(r) 1_H$, then applying λ to $\varepsilon(r) 1_H = v^{-1}(r^{(1)} \otimes 1_R) r_{(-1)}^{(2)} v(r_{(0)}^{(2)} \otimes 1_R)$ and using the fact that $\lambda \Psi(v) = \varepsilon_C$, we obtain $v^{-1}(r \otimes 1) = \varepsilon(r)$ so that $v(r \otimes 1) = \varepsilon(r)$ also.

The argument for elements $1 \otimes r$ is the same. \square

The propositions above now prove Theorem 4.3.

Proof. (of Theorem 4.3.) Theorem 4.1 and Propositions 4.5, 4.6, 4.7, 4.8 show that $V = G(\Xi)$ is the set of $v \in \mathcal{G}$ satisfying (27) through (30). \square

REMARK 4.9. In general the map $v = F(\xi)$ in Theorem 4.3, is not a cocycle [ABM, Example 5.11, Remarks 5.13(ii)] although in these cases (R^v, m^v, u) is an associative algebra. See also [ABM1] for a discussion of when $F(\xi)$ is a cocycle. In general, it is unknown whether (R^v, m^v, u) is an associative algebra or not.

PROPOSITION 4.10. *Let $\xi \in \Xi$ and $v := G(\xi)$ as above. Then the following are equivalent:*

- (i) $v \in Z_H^2(R, K)$;
- (ii) $\partial_R^2(v) = \varepsilon_D$ with $D := R \otimes R \otimes R$;
- (iii) $\lambda \circ [\Psi(\partial_R^2(v))] = \varepsilon_D$;
- (iv) $\partial_R^2(v) = \varepsilon_D$ on ${}^{co(H)}(D)$.

Proof. Set $A := R \#_\xi H$. Since $\partial_R^2(v) = \varepsilon_D$ if and only if $v \in \text{Ker}(\partial_R^2) = \Omega^2(\text{Ker}(\partial_A^2)) = \Omega^2(Z_H^2(A, K))$, the equivalence of (i) and (ii) follows from the definition of $Z_H^2(R, K)$ from [ABM] (or see Section 3.2). To see the equivalence of (ii) and (iii), apply λ to both sides of (29) to obtain $\lambda \circ \Psi(\partial_R^2(v)) = \partial_R^2(v)$.

Since (ii) implies (iv) is trivial, it remains to show that (iv) implies (iii).

Note that for $z \in D$,

$$(\lambda \circ [\Psi(\partial_R^2(v))])(z) = \lambda(z_{(-1)} \partial_R^2(v)(z_{(0)})) = \lambda(z_{(-1)}) \partial_R^2(v)(z_{(0)}) = \partial_R^2(v)(\lambda(z_{(-1)}) z_{(0)}).$$

To complete the proof it suffices to check that $\lambda(z_{(-1)}) z_{(0)} \in {}^{co(H)}D$. Indeed, we have

$$\rho(\lambda(z_{(-1)}) z_{(0)}) = \lambda(z_{(-2)}) z_{(-1)} \otimes z_{(0)} = 1_H \lambda(z_{(-1)}) \otimes z_{(0)} = 1_H \otimes \lambda(z_{(-1)}) z_{(0)}.$$

\square

5. THE DUAL QUASI-BIALGEBRA A^{vA}

In the previous section, for (R, ξ) a pre-bialgebra with cocycle in ${}^H_H \mathcal{YD}$, the defining properties of ξ were translated to properties of $v := G(\xi) \in \text{Reg}_H(R \otimes R, K)$. In other words, we showed that the functor from \mathcal{R} to \mathcal{R}' which takes an object (R, ξ) to $(R, v := (\lambda\xi)^{-1})$ is an isomorphism. In this section we will first show that R^v , the pre-bialgebra R with its multiplication twisted by v , is a dual quasi-bialgebra in ${}^H_H \mathcal{YD}$ with reassociator $\partial_R^2(v)$, in other words that T_1 is a functor. First we recall the definitions of dual quasi-bialgebra in vector spaces and in ${}^H_H \mathcal{YD}$ and define a process of bosonization taking dual quasi-bialgebras in ${}^H_H \mathcal{YD}$ to dual quasi-bialgebras in vector spaces over K .

5.1. Dual quasi-bialgebras and bosonization. Recall from [Maj, page 66] that a *dual quasi-bialgebra* $(D, m, u, \Delta, \varepsilon, \alpha)$ is a coalgebra (D, Δ, ε) with coalgebra homomorphisms $m : D \otimes D \rightarrow D$ and $u : K \rightarrow D$ ($1_D := u(1_K)$) such that $1_D x = x = x 1_D$ for all $x \in D$ and $\alpha \in \text{Reg}(D^{\otimes 3}, K)$ is such that $\partial_D^3(\alpha) = \varepsilon_{D \otimes D \otimes D}$, α is unital and m is α -associative. Equivalently,

- (i) $\alpha(D \otimes D \otimes m) * \alpha(m \otimes D \otimes D) = (\varepsilon \otimes \alpha) * \alpha(D \otimes m \otimes D) * (\alpha \otimes \varepsilon)$
- (ii) $\alpha(D \otimes 1_D \otimes D) = \alpha(1_D \otimes D \otimes D) = \alpha(D \otimes D \otimes 1_D) = \varepsilon_{D \otimes D}$,
- (iii) $m(D \otimes m) * \alpha = \alpha * m(m \otimes D)$.

Note that in (ii) any of the three equalities such as $\alpha(1_D \otimes D \otimes D) = \varepsilon_{D \otimes D}$ implies that α is unital. The map α is called the reassociator.

A unital map $v \in \text{Reg}(D \otimes D, K)$ is called a *gauge transformation*. Then the twisted dual quasi-bialgebra $D^v = (D, m^v := v * m * v^{-1}, u, \Delta, \varepsilon, \alpha_{D^v})$ is also a dual quasi-bialgebra where the reassociator α_{D^v} is defined by:

$$(32) \quad \alpha_{D^v} := (\varepsilon \otimes v) * v(D \otimes m) * \alpha * v^{-1}(m \otimes D) * (v^{-1} \otimes \varepsilon).$$

Note that, whenever α is trivial, one has $\alpha_{D^v} = \partial_D^2(v)$, in the sense of Subsection 3.1.

A morphism of dual quasi-bialgebras $f : (D, m, u, \Delta, \varepsilon, \alpha) \rightarrow (D', m', u', \Delta', \varepsilon', \alpha')$ is a coalgebra homomorphism $f : (D, \Delta, \varepsilon) \rightarrow (D', \Delta', \varepsilon')$ such that $m'(f \otimes f) = fm$, $fu = u'$ and $\alpha'(f \otimes f \otimes f) = \alpha$. It is an isomorphism of quasi-bialgebras if, in addition, it is invertible.

DEFINITION 5.1. Dual quasi-bialgebras A and B are called *quasi-isomorphic* (or equivalent) whenever $A \cong B^v$ as dual quasi-bialgebras for some gauge transformation $v \in (B \otimes B)^*$.

We now give the definition of a dual quasi-bialgebra Q in ${}^H_H\mathcal{YD}$ and show that a K -dual quasi-bialgebra can be constructed from Q by bosonization. Although our purpose in the end is to study Hopf algebras whose coradicals are semisimple sub-Hopf algebras, this result is interesting on its own and adds to the literature on constructions with dual quasi-bialgebras. (See, for example, [BN, Section 3] where a smash product $B \# H$ with a quasi-Hopf structure is studied for H a quasi-Hopf algebra and B a braided Hopf algebra in ${}^H_H\mathcal{YD}$.)

DEFINITION 5.2. Let H be a Hopf algebra. A *dual quasi-bialgebra* $(Q, m, u, \Delta, \varepsilon, \alpha)$ in the braided monoidal category ${}^H_H\mathcal{YD}$ is a coalgebra (Q, Δ, ε) in ${}^H_H\mathcal{YD}$ together with coalgebra homomorphisms $m : Q \otimes Q \rightarrow Q$ and $u : K \rightarrow Q$ in ${}^H_H\mathcal{YD}$ and a convolution invertible element $\alpha \in {}^H_H\mathcal{YD}(Q^{\otimes 3}, K)$ (braided reassociator) such that

$$(33) \quad \alpha(Q \otimes Q \otimes m) * \alpha(m \otimes Q \otimes Q) = (\varepsilon \otimes \alpha) * \alpha(Q \otimes m \otimes Q) * (\alpha \otimes \varepsilon),$$

$$(34) \quad \alpha(Q \otimes u \otimes Q) = \alpha(u \otimes Q \otimes Q) = \alpha(Q \otimes Q \otimes u) = \varepsilon_{Q \otimes Q},$$

$$(35) \quad m(Q \otimes m) * \alpha = \alpha * m(m \otimes Q),$$

$$(36) \quad m(u \otimes Q) = \text{Id}_Q = m(Q \otimes u).$$

Note that in (34) any of the three equalities such as $\alpha(u \otimes Q \otimes Q) = \varepsilon_{Q \otimes Q}$ implies that α is unital.

The next result proves there is a dual quasi-bialgebra associated to any dual quasi-bialgebra Q in the braided monoidal category ${}^H_H\mathcal{YD}$.

PROPOSITION 5.3. Let H be a Hopf algebra and let $(Q, m, u, \Delta, \varepsilon, \alpha)$ be a dual quasi-bialgebra in ${}^H_H\mathcal{YD}$. Set $B := Q \otimes H$. Then

$$Q \# H := (B, m_B, u_B, \Delta_B, \varepsilon_B, \alpha_B)$$

is an ordinary dual quasi-bialgebra where $Q \# H$ has the usual coalgebra structure and multiplication and unit maps namely

$$\begin{aligned} m_B(r \# h \otimes s \# l) &:= m(r \otimes h_{(1)}s) \# h_{(2)}l, & u_B(1_K) &:= 1_Q \# 1_H, \\ \Delta_B(r \# h) &:= (r^{(1)} \# r_{(-1)}^{(2)} h_{(1)}) \otimes (r_{(0)}^{(2)} \# h_{(2)}), & \varepsilon_B(r \# h) &:= \varepsilon(r) \varepsilon_H(h), \end{aligned}$$

and the reassociator is given by $\alpha_B = \mathcal{U}_{H,Q}^3(\alpha)$ namely

$$\alpha_B(r \# h \otimes s \# l \otimes t \# k) := \alpha(r \otimes h_{(1)}s \otimes h_{(2)}lt) \varepsilon_H(k).$$

Proof. Since (Q, Δ, ε) is a coalgebra in ${}^H_H\mathcal{YD}$, $(B, \Delta_B, \varepsilon_B)$ is the smash coproduct of Q by H . It is straightforward to verify that $u_B : K \rightarrow B$ is a coalgebra map and that $\varepsilon_B m_B(r \# h \otimes s \# l) = \varepsilon(r) \varepsilon_H(h) \varepsilon(s) \varepsilon_H(l)$; these are left to the reader. Similarly it is clear that $m_B(1_B \otimes s \# l) = s \# l$ and also $m_B(r \# h \otimes 1_B) = m(r \otimes h_{(1)}1_Q) \# h_{(2)} = m(r \otimes 1_Q) \# h = r \# h$. The map m_B is clearly right H -linear and is also left H -linear since

$$\begin{aligned} km_B(r \# h \otimes s \# l) &= k_{(1)}m(r \otimes h_{(1)}s) \# k_{(2)}h_{(2)}l = m(k_{(1)}r \otimes k_{(2)}h_{(1)}s) \# k_{(3)}h_{(2)}l \\ &= m_B(k_{(1)}r \# k_{(2)}h \otimes s \# l) = m_B(k(r \# h \otimes s \# l)). \end{aligned}$$

Furthermore, m_B is H -balanced since

$$\begin{aligned} m_B((r \# h)k \otimes s \# l) &= m_B(r \# hk \otimes s \# l) = m(r \otimes h_{(1)}k_{(1)}s) \# h_{(2)}k_{(2)}l \\ &= m_B(r \# h \otimes k_{(1)}s \# k_{(2)}l) = m_B(r \# h \otimes k(s \# l)). \end{aligned}$$

We check next that $m_B : B \otimes B \rightarrow B$ is a coalgebra homomorphism. Since Δ_B is H -bilinear and m_B is H -bilinear and H -balanced, it suffices to show that m_B is a coalgebra homomorphism

on elements of the form $r\#1_H \otimes s\#1_H$. In the third step we use the fact that $m : Q \otimes Q \rightarrow Q$ is a coalgebra map and $\varepsilon_B m_B = \varepsilon_{B \otimes B}$.

$$\begin{aligned}
\Delta_B m_B (r\#1_H \otimes s\#1_H) &= \Delta_B [m(r \otimes s)\#1_H] \\
&= [m(r \otimes s)]^{(1)} \# [m(r \otimes s)]^{(2)}_{\langle -1 \rangle} \otimes [m(r \otimes s)]^{(2)}_{\langle 0 \rangle} \# 1_H \\
&= m(r^{(1)} \otimes r^{(2)}_{\langle -1 \rangle} s^{(1)}) \# [m(r^{(2)}_{\langle 0 \rangle} \otimes s^{(2)})]_{\langle -1 \rangle} \otimes [m(r^{(2)}_{\langle 0 \rangle} \otimes s^{(2)})]_{\langle 0 \rangle} \# 1_H \\
&= m(r^{(1)} \otimes r^{(2)}_{\langle -1 \rangle} s^{(1)}) \# (r^{(2)}_{\langle 0 \rangle} \otimes s^{(2)})_{\langle -1 \rangle} \otimes m[(r^{(2)}_{\langle 0 \rangle} \otimes s^{(2)})_{\langle 0 \rangle}] \# 1_H \\
&= m(r^{(1)} \otimes r^{(2)}_{\langle -2 \rangle} s^{(1)}) \# r^{(2)}_{\langle -1 \rangle} s^{(2)}_{\langle -1 \rangle} \otimes m(r^{(2)}_{\langle 0 \rangle} \otimes s^{(2)}_{\langle 0 \rangle}) \# 1_H \\
&= m \left(r^{(1)} \otimes r^{(2)}_{\langle -2 \rangle} s^{(1)} \right) \# r^{(2)}_{\langle -1 \rangle} s^{(2)}_{\langle -1 \rangle} \otimes m_B \left[r^{(2)}_{\langle 0 \rangle} \# 1_H \otimes s^{(2)}_{\langle 0 \rangle} \# 1_H \right] \\
&= m_B \left[r^{(1)} \# r^{(2)}_{\langle -1 \rangle} \otimes s^{(1)} \# s^{(2)}_{\langle -1 \rangle} \right] \otimes m_B \left[r^{(2)}_{\langle 0 \rangle} \# 1_H \otimes s^{(2)}_{\langle 0 \rangle} \# 1_H \right] \\
&= m_B \left[(r\#1_H)_{(1)} \otimes (s\#1_H)_{(1)} \right] \otimes m_B \left[(r\#1_H)_{(2)} \otimes (s\#1_H)_{(2)} \right].
\end{aligned}$$

It is clear that $\alpha_B = \mathcal{U}_{H,R}^3(\alpha)$ is convolution invertible with inverse $\mathcal{U}_{H,R}^3(\alpha^{-1})$.

If ϕ is any H -multibalanced, H -bilinear map from $B \otimes B \otimes B$ to B , then

$$\phi(r\#h \otimes s\#l \otimes t\#k) = \phi(r\#1_H \otimes h_{(1)}s\#1_H \otimes h_{(2)}l_{(1)}t\#1_H) h_{(3)}l_{(2)}k.$$

Since m_B is H -balanced and H -bilinear, then $m_B^l := m_B(m_B \otimes B)$ and $m_B^r := m_B(B \otimes m_B)$ are H -multibalanced and H -bilinear too and so are $m_B^r * \alpha_B$ and $\alpha_B * m_B^l$. Thus it suffices to check that $m_B^r * \alpha_B = \alpha_B * m_B^l$, i.e., m_B is associative up to multiplication with the reassociator, on elements of the form $r\#1_H \otimes s\#1_H \otimes t\#1_H$ for $r, s, t \in R$:

$$\begin{aligned}
&[m_B^r * \alpha_B](r\#1_H \otimes s\#1_H \otimes t\#1_H) \\
&= m_B^r(r^{(1)} \# r^{(2)}_{\langle -1 \rangle} \otimes s^{(1)} \# s^{(2)}_{\langle -1 \rangle} \otimes t^{(1)} \# t^{(2)}_{\langle -1 \rangle}) \alpha_B(r^{(2)}_{\langle 0 \rangle} \# 1_H \otimes s^{(2)}_{\langle 0 \rangle} \# 1_H \otimes t^{(2)}_{\langle 0 \rangle} \# 1_H) \\
&= m_B^r(r^{(1)} \# r^{(2)}_{\langle -1 \rangle} \otimes s^{(1)} \# s^{(2)}_{\langle -1 \rangle} \otimes t^{(1)} \# t^{(2)}_{\langle -1 \rangle}) \alpha(r^{(2)}_{\langle 0 \rangle} \otimes s^{(2)}_{\langle 0 \rangle} \otimes t^{(2)}_{\langle 0 \rangle}) \\
&= m_B^r(r^{(1)} \# 1 \otimes r^{(2)}_{\langle -3 \rangle} s^{(1)} \# 1 \otimes r^{(2)}_{\langle -2 \rangle} s^{(2)}_{\langle -2 \rangle} t^{(1)} \# 1) r^{(2)}_{\langle -1 \rangle} s^{(2)}_{\langle -1 \rangle} t^{(2)}_{\langle -1 \rangle} \alpha(r^{(2)}_{\langle 0 \rangle} \otimes s^{(2)}_{\langle 0 \rangle} \otimes t^{(2)}_{\langle 0 \rangle}) \\
&= m_B(r^{(1)} \# 1 \otimes m(r^{(2)}_{\langle -3 \rangle} s^{(1)} \otimes r^{(2)}_{\langle -2 \rangle} s^{(2)}_{\langle -2 \rangle} t^{(1)}) \# 1) r^{(2)}_{\langle -1 \rangle} s^{(2)}_{\langle -1 \rangle} t^{(2)}_{\langle -1 \rangle} \alpha(r^{(2)}_{\langle 0 \rangle} \otimes s^{(2)}_{\langle 0 \rangle} \otimes t^{(2)}_{\langle 0 \rangle}) \\
&= [m(r^{(1)} \otimes m(r^{(2)}_{\langle -3 \rangle} s^{(1)} \otimes r^{(2)}_{\langle -2 \rangle} s^{(2)}_{\langle -2 \rangle} t^{(1)}))] \# r^{(2)}_{\langle -1 \rangle} s^{(2)}_{\langle -1 \rangle} t^{(2)}_{\langle -1 \rangle} \alpha(r^{(2)}_{\langle 0 \rangle} \otimes s^{(2)}_{\langle 0 \rangle} \otimes t^{(2)}_{\langle 0 \rangle}) \\
&= m(Q \otimes m) \left[r^{(1)} \otimes r^{(2)}_{\langle -2 \rangle} s^{(1)} \otimes r^{(2)}_{\langle -1 \rangle} s^{(2)}_{\langle -1 \rangle} t^{(1)} \right] \# \Psi(\alpha) \left(r^{(2)}_{\langle 0 \rangle} \otimes s^{(2)}_{\langle 0 \rangle} \otimes t^{(2)}_{\langle 0 \rangle} \right) \\
&\stackrel{(29)}{=} m(Q \otimes m) \left[r^{(1)} \otimes r^{(2)}_{\langle -2 \rangle} s^{(1)} \otimes r^{(2)}_{\langle -1 \rangle} s^{(2)}_{\langle -1 \rangle} t^{(1)} \right] \# \alpha \left(r^{(2)}_{\langle 0 \rangle} \otimes s^{(2)}_{\langle 0 \rangle} \otimes t^{(2)}_{\langle 0 \rangle} \right) \\
&= [m(Q \otimes m) * \alpha](r \otimes s \otimes t) \# 1_H \\
&\stackrel{(35)}{=} [\alpha * m(m \otimes Q)](r \otimes s \otimes t) \# 1_H \\
&= \alpha \left(r^{(1)} \otimes r^{(2)}_{\langle -2 \rangle} s^{(1)} \otimes r^{(2)}_{\langle -1 \rangle} s^{(2)}_{\langle -1 \rangle} t^{(1)} \right) m(m \otimes Q) \left(r^{(2)}_{\langle 0 \rangle} \otimes s^{(2)}_{\langle 0 \rangle} \otimes t^{(2)}_{\langle 0 \rangle} \right) \# 1_H \\
&= \alpha \left(r^{(1)} \otimes r^{(2)}_{\langle -2 \rangle} s^{(1)} \otimes r^{(2)}_{\langle -1 \rangle} s^{(2)}_{\langle -1 \rangle} t^{(1)} \right) m \left[m \left(r^{(2)}_{\langle 0 \rangle} \otimes s^{(2)}_{\langle 0 \rangle} \right) \otimes t^{(2)}_{\langle 0 \rangle} \right] \# 1_H \\
&= \alpha \left(r^{(1)} \otimes r^{(2)}_{\langle -2 \rangle} s^{(1)} \otimes r^{(2)}_{\langle -1 \rangle} s^{(2)}_{\langle -1 \rangle} t^{(1)} \right) m_B \left(m \left(r^{(2)}_{\langle 0 \rangle} \otimes s^{(2)}_{\langle 0 \rangle} \right) \# 1_H \otimes t^{(2)}_{\langle 0 \rangle} \# 1_H \right) \\
&= \alpha \left(r^{(1)} \otimes r^{(2)}_{\langle -2 \rangle} s^{(1)} \otimes r^{(2)}_{\langle -1 \rangle} s^{(2)}_{\langle -1 \rangle} t^{(1)} \right) m_B^l \left(r^{(2)}_{\langle 0 \rangle} \# 1_H \otimes s^{(2)}_{\langle 0 \rangle} \# 1_H \otimes t^{(2)}_{\langle 0 \rangle} \# 1_H \right) \\
&= \alpha_B \left(r^{(1)} \# r^{(2)}_{\langle -1 \rangle} \otimes s^{(1)} \# s^{(2)}_{\langle -1 \rangle} \otimes t^{(1)} \# t^{(2)}_{\langle -1 \rangle} \right) m_B^l \left(r^{(2)}_{\langle 0 \rangle} \# 1_H \otimes s^{(2)}_{\langle 0 \rangle} \# 1_H \otimes t^{(2)}_{\langle 0 \rangle} \# 1_H \right) \\
&= \alpha_B \left((r\#1_H)_{(1)} \otimes (s\#1_H)_{(1)} \otimes (t\#1_H)_{(1)} \right) m_B^l \left((r\#1_H)_{(2)} \otimes (s\#1_H)_{(2)} \otimes (t\#1_H)_{(2)} \right) \\
&= (\alpha_B * m_B^l)(r\#1_H \otimes s\#1_H \otimes t\#1_H).
\end{aligned}$$

It remains to prove the cocycle condition for α_B . First we note that the following equalities can be easily checked by applying $\Omega_{H,Q}^4$ to both sides.

$$(37) \quad \alpha_B(B \otimes B \otimes m_B) = \mathcal{U}_{H,Q}^4[\alpha(Q \otimes Q \otimes m)],$$

$$(38) \quad \alpha_B(m_B \otimes B \otimes B) = \mathcal{U}_{H,Q}^4[\alpha(m \otimes Q \otimes Q)],$$

$$(39) \quad \alpha_B(B \otimes m_B \otimes B) = \mathcal{U}_{H,Q}^4(\alpha(Q \otimes m \otimes Q)),$$

$$(40) \quad \varepsilon_B \otimes \alpha_B = \mathcal{U}_{H,Q}^4(\varepsilon \otimes \alpha), \text{ and } \alpha_B \otimes \varepsilon_B = \mathcal{U}_{H,Q}^4(\alpha \otimes \varepsilon).$$

Therefore we get

$$\begin{aligned} & \alpha_B(B \otimes B \otimes m_B) * \alpha_B(m_B \otimes B \otimes B) \\ \stackrel{(37),(38)}{=} & \mathcal{U}_{H,Q}^4[\alpha(Q \otimes Q \otimes m)] * \mathcal{U}_{H,Q}^4[\alpha(m \otimes Q \otimes Q)] \\ = & \mathcal{U}_{H,Q}^4[\alpha(Q \otimes Q \otimes m) * \alpha(m \otimes Q \otimes Q)] \\ \stackrel{(33)}{=} & \mathcal{U}_{H,Q}^4[(\varepsilon \otimes \alpha) * \alpha(Q \otimes m \otimes Q) * (\alpha \otimes \varepsilon)] \\ = & \mathcal{U}_{H,Q}^4(\varepsilon \otimes \alpha) * \mathcal{U}_{H,Q}^4(\alpha(Q \otimes m \otimes Q)) * \mathcal{U}_{H,Q}^4(\alpha \otimes \varepsilon) \\ \stackrel{(39),(40)}{=} & (\varepsilon_B \otimes \alpha_B) * \alpha_B(B \otimes m_B \otimes B) * (\alpha_B \otimes \varepsilon_B) \end{aligned}$$

Hence we have proved the cocycle condition for α_B as desired. \square

DEFINITION 5.4. Let H be a Hopf algebra and let $(Q, m, u, \Delta, \varepsilon, \alpha)$ be a dual quasi-bialgebra in ${}^H_H\mathcal{YD}$. Set $B := Q \otimes H$. Then

$$Q \# H := (B, m_B, u_B, \Delta_B, \varepsilon_B, \alpha_B)$$

will be called the *bosonization* of the dual quasi-bialgebra $(Q, m, u, \Delta, \varepsilon, \alpha)$ in ${}^H_H\mathcal{YD}$ by H .

If Q is connected, by the same proof as [AMStu, Theorem 3.9], the coradical of $Q \# H$ is $K \otimes H$. Thus we have shown that the second bosonization functor B_2 mentioned in the Introduction actually maps objects in \mathcal{Q} to objects in \mathcal{B} . The fact that B_2 preserves morphisms is straightforward and so we omit it.

5.2. The dual quasi-bialgebra from a pre-bialgebra with cocycle. In this section we show that if (R, ξ) is a pre-bialgebra with cocycle, then for $v := G(\xi)$ with G the map from Theorem 4.3, twisting R by v gives R^v the structure of a dual quasi-bialgebra in ${}^H_H\mathcal{YD}$. In other words, we show that the first twisting functor T_1 from the introduction maps objects in \mathcal{R} to objects in \mathcal{Q} . Again, we leave the verification that T_1 also preserves morphisms to the reader.

PROPOSITION 5.5. Let (R, ξ) be a connected pre-bialgebra with cocycle in ${}^H_H\mathcal{YD}$. Let $v := G(\xi) = (\lambda\xi)^{-1}$. Then $R^v := (R, m^v, u, \Delta, \varepsilon, \alpha := \partial_R^2(v))$ is a connected dual quasi-bialgebra in ${}^H_H\mathcal{YD}$.

Proof. By construction, (R, Δ, ε) is a connected coalgebra in ${}^H_H\mathcal{YD}$ and m^v and u are coalgebra homomorphisms. From Corollary 3.4,

$$\alpha := \partial_R^2(v) = v(R \otimes m^v) * (\varepsilon \otimes v) * (v^{-1} \otimes \varepsilon) * v^{-1}(m^v \otimes R),$$

so that α is convolution invertible and by Remark 4.4, α is in ${}^H_H\mathcal{YD}$. It remains to check that α satisfies (33) through (36). It is straightforward to show that unitality of v, v^{-1} implies (34), the unital property for α . Theorem 4.3 implies that m^v satisfies (35). It is straightforward to prove (36). It remains to verify (33). To simplify notation in the following computation, we set:

$$\begin{aligned} m_v^1 &:= m^v \otimes R \otimes R, \quad m_v^2 := R \otimes m^v \otimes R, \quad m_v^3 := R \otimes R \otimes m^v; \\ m_v^r &:= m^v(R \otimes m^v), \quad m_v^l := m^v(m^v \otimes R); \\ \alpha_+ &:= (\partial_R^2)_+(v) = v(R \otimes m^v) * (\varepsilon \otimes v), \quad \alpha_- := (\partial_R^2)_-(v^{-1}) = (v^{-1} \otimes \varepsilon) * v^{-1}(m^v \otimes R). \end{aligned}$$

Now note that by (6),

$$\begin{aligned} \alpha m_v^3 &= v(R \otimes m^v) m_v^3 * (\varepsilon \otimes v) m_v^3 * (v^{-1} \otimes \varepsilon) m_v^3 * v^{-1}(m^v \otimes R) m_v^3 \\ &= v(R \otimes m_v^r) * (\varepsilon \otimes v(R \otimes m^v)) * (v^{-1} \otimes \varepsilon_{R \otimes R}) * v^{-1}(m^v \otimes m^v) \text{ and} \end{aligned}$$

$$\begin{aligned}
\alpha m_v^1 &= v(R \otimes m^v) m_v^1 * (\varepsilon \otimes v) m_v^1 * (v^{-1} \otimes \varepsilon) m_v^1 * v^{-1}(m^v \otimes R) m_v^1 \\
&= v(m^v \otimes m^v) * (\varepsilon_{R \otimes R} \otimes v) * (v^{-1}(m^v \otimes R) \otimes \varepsilon_R) * v^{-1}(m_v^l \otimes R),
\end{aligned}$$

so that $\alpha m_v^3 * \alpha m_v^1$, the left hand side of (33), is equal to

$$v(R \otimes m_v^r) * (\varepsilon \otimes v(R \otimes m^v)) * (v^{-1} \otimes v) * (v^{-1}(m^v \otimes R) \otimes \varepsilon) * v^{-1}(m_v^l \otimes R).$$

Since

$$(\varepsilon \otimes v(R \otimes m^v)) * (\varepsilon \otimes \varepsilon \otimes v) = \varepsilon \otimes [v(R \otimes m^v) * (\varepsilon \otimes v)] = \varepsilon \otimes \alpha_+,$$

and

$$(v^{-1} \otimes \varepsilon \otimes \varepsilon) * (v^{-1}(m^v \otimes R) \otimes \varepsilon) = [(v^{-1} \otimes \varepsilon) * (v^{-1}(m^v \otimes R))] \otimes \varepsilon = \alpha_- \otimes \varepsilon,$$

then $\alpha m_v^3 * \alpha m_v^1$ equals:

$$\begin{aligned}
&v(R \otimes m_v^r) * (\varepsilon \otimes \alpha_+) * (\alpha_- \otimes \varepsilon) * v^{-1}(m_v^l \otimes R) \\
&\stackrel{(8)}{=} v(R \otimes m_v^r * \alpha_+) * (\alpha_- \otimes \varepsilon) * v^{-1}(m_v^l \otimes R) \\
&= v(R \otimes m_v^r * \alpha * (\alpha_-)^{-1}) * (\alpha_- \otimes \varepsilon) * v^{-1}(m_v^l \otimes R) \\
&\stackrel{(35)}{=} v(R \otimes \alpha * m_v^l * (\alpha_-)^{-1}) * (\alpha_- \otimes \varepsilon) * v^{-1}(m_v^l \otimes R) \\
&\stackrel{(8)}{=} (\varepsilon \otimes \alpha) * v(R \otimes m_v^l) * (\varepsilon \otimes (\alpha_-)^{-1}) * (\alpha_- \otimes \varepsilon) * v^{-1}(m_v^l \otimes R) \\
&= (\varepsilon \otimes \alpha) * v(R \otimes m^v) m_v^2 * (\varepsilon \otimes (\alpha_-)^{-1}) * (\alpha_- \otimes \varepsilon) * v^{-1}(m_v^l \otimes R) \\
&= (\varepsilon_R \otimes \alpha) * [\alpha_+ * (\varepsilon \otimes v^{-1})] m_v^2 * (\varepsilon \otimes (\alpha_-)^{-1}) * (\alpha_- \otimes \varepsilon) * v^{-1}(m_v^l \otimes R) \\
&\stackrel{(6)}{=} (\varepsilon \otimes \alpha) * \alpha_+ m_v^2 * (\varepsilon \otimes v^{-1}) m_v^2 * (\varepsilon \otimes (\alpha_-)^{-1}) * (\alpha_- \otimes \varepsilon) * v^{-1}(m_v^l \otimes R)
\end{aligned}$$

Moreover

$$\begin{aligned}
&(\varepsilon \otimes v^{-1}) m_v^2 * (\varepsilon \otimes (\alpha_-)^{-1}) * (\alpha_- \otimes \varepsilon) * v^{-1}(m_v^l \otimes R) \\
&= (\varepsilon \otimes v^{-1}(m^v \otimes R)) * (\varepsilon \otimes (\alpha_-)^{-1}) * (\alpha_- \otimes \varepsilon) * v^{-1}(m_v^l \otimes R) \\
&= (\varepsilon \otimes [v^{-1}(m_v \otimes R) * (\alpha_-)^{-1}]) * (\alpha_- \otimes \varepsilon) * v^{-1}(m_v^l \otimes R) \\
&= (\varepsilon \otimes v \otimes \varepsilon) * (\alpha_- \otimes \varepsilon) * v^{-1}(m_v^l \otimes R) \\
&\stackrel{(9)}{=} (\varepsilon \otimes v \otimes \varepsilon) * v^{-1}([\alpha_- * m_v^l] \otimes R) \\
&\stackrel{(9)}{=} v^{-1}([\varepsilon \otimes v] * \alpha_- * m_v^l) \otimes R \\
&= v^{-1}([\varepsilon \otimes v] * (\alpha_+)^{-1} * \alpha_+ * \alpha_- * m_v^l) \otimes R \\
&= v^{-1}([\varepsilon \otimes v] * (\alpha_+)^{-1} * \alpha * m_v^l) \otimes R \\
&\stackrel{(35)}{=} v^{-1}([\varepsilon \otimes v] * (\alpha_+)^{-1} * m_v^r * \alpha) \otimes R \\
&= v^{-1}([v^{-1}(R \otimes m^v) * m_v^r * \alpha] \otimes R) \\
&\stackrel{(9)}{=} (v^{-1}(R \otimes m^v) \otimes \varepsilon) * v^{-1}([m_v^r * \alpha] \otimes R) \\
&\stackrel{(10)}{=} (v^{-1}(R \otimes m^v) \otimes \varepsilon) * v^{-1}(m_v^r \otimes R) * (\alpha \otimes \varepsilon) \\
&= [(v^{-1} \otimes \varepsilon) m_v^2] * [v^{-1}(m^v \otimes R) m_v^2] * (\alpha \otimes \varepsilon) \\
&\stackrel{(6)}{=} [\alpha_- m_v^2] * (\alpha \otimes \varepsilon).
\end{aligned}$$

Thus

$$(\varepsilon \otimes v^{-1}) m_v^2 * (\varepsilon \otimes (\alpha_-)^{-1}) * (\alpha_- \otimes \varepsilon) * v^{-1}(m_v^l \otimes R) = (\alpha_- m_v^2) * (\alpha \otimes \varepsilon),$$

and so

$$\begin{aligned}
\alpha m_v^3 * \alpha m_v^1 &= (\varepsilon \otimes \alpha) * \alpha_+ m_v^2 * (\varepsilon \otimes v^{-1}) m_v^2 * (\varepsilon \otimes (\alpha_-)^{-1}) * (\alpha_- \otimes \varepsilon) * v^{-1}(m_v^l \otimes R) \\
&= (\varepsilon \otimes \alpha) * (\alpha_+ m_v^2) * (\alpha_- m_v^2) * (\alpha \otimes \varepsilon) \stackrel{(6)}{=} (\varepsilon \otimes \alpha) * \alpha m_v^2 * (\alpha \otimes \varepsilon),
\end{aligned}$$

and α satisfies the 3-cocycle condition. \square

REMARK 5.6. It is unknown whether $(R, m^v, u, \Delta, \varepsilon)$ is a braided bialgebra in the braided monoidal category ${}^H_H\mathcal{YD}$, see Remark 4.9.

5.3. The bosonization of R^v with H . Let (R, ξ) be a connected pre-bialgebra with cocycle in ${}^H_H\mathcal{YD}$, and let $A := R \#_\xi H$. Since the coalgebra A is the smash coproduct of R with H , comultiplication is given by (19). It is useful to have:

$$(41) \quad \Delta_A^2(r \# h) = (r^{(1)} \# r_{\langle -1 \rangle}^{(2)} r_{\langle -2 \rangle}^{(3)} h_{(1)}) \otimes (r_{\langle 0 \rangle}^{(2)} \# r_{\langle -1 \rangle}^{(3)} h_{(2)}) \otimes (r_{\langle 0 \rangle}^{(3)} \# h_{(3)}).$$

Let $v := G(\xi)$ as in the preceding sections and let $v_A := \mathcal{U}^2(v)$ so that $v_A(x \# h \otimes y \# h') = v(x \otimes hy) \varepsilon_H(h')$. Since v is unital, v_A is also and thus is a gauge transformation.

In this subsection, we prove that the twisting of A by v_A is the bosonization of the dual quasi-bialgebra R^v and H defined in Proposition 5.3, i.e., that $A^{v_A} = R^v \# H$. Since in general v_A might not be a cocycle, we cannot say that A^{v_A} is a bialgebra, but A^{v_A} is always a dual quasi-bialgebra.

PROPOSITION 5.7. *For $(R, m, u, \Delta, \varepsilon, \xi)$ as above, let $A := R \#_\xi H$, let $v := G(\xi)$ and let $v_A := \mathcal{U}^2(v)$. Then*

$$A^{v_A} = R^v \# H,$$

the bosonization of the dual quasi-bialgebra $(R, m^v, u, \Delta, \varepsilon, \partial_R^2(v))$ in ${}^H_H\mathcal{YD}$ by H .

Proof. Since $u_{A^{v_A}}(1_K) = u_A(1_K) = 1_R \otimes 1_H = 1_{R^v} \otimes 1 = u_{R^v \# H}(1_K)$, the unit maps for A^{v_A} and $R^v \# H$ are the same. It remains to show that the multiplication maps m^{v_A} and $m_{R^v \# H}$ are the same on $R \# H$, and that the reassociators coincide.

We begin by computing the product in A^v of two elements from $R \# 1_H$. As usual, when the context is clear, we omit the subscript H from 1_H , and write m instead of m_R .

$$\begin{aligned} & m^{v_A}(r \# 1 \otimes s \# 1) \\ &= (v_A * m_A * v_A^{-1})(r \# 1 \otimes s \# 1) \\ &= v_A[(r \# 1)_{(1)} \otimes (s \# 1)_{(1)}] m_A[(r \# 1)_{(2)} \otimes (s \# 1)_{(2)}] v_A^{-1}[(r \# 1)_{(3)} \otimes (s \# 1)_{(3)}] \\ &\stackrel{(41)}{=} v_A[r^{(1)} \# r_{\langle -1 \rangle}^{(2)} r_{\langle -2 \rangle}^{(3)} \otimes s^{(1)} \# s_{\langle -1 \rangle}^{(2)} s_{\langle -2 \rangle}^{(3)}] m_A[r_{\langle 0 \rangle}^{(2)} \# r_{\langle -1 \rangle}^{(3)} \otimes s_{\langle 0 \rangle}^{(2)} \# s_{\langle -1 \rangle}^{(3)}] v_A^{-1}[r_{\langle 0 \rangle}^{(3)} \# 1 \otimes s_{\langle 0 \rangle}^{(3)} \# 1] \\ &= v[r^{(1)} \otimes r_{\langle -1 \rangle}^{(2)} r_{\langle -2 \rangle}^{(3)} s^{(1)}] m_A[r_{\langle 0 \rangle}^{(2)} \# r_{\langle -1 \rangle}^{(3)} \otimes s^{(2)} \# s_{\langle -1 \rangle}^{(3)}] v^{-1}[r_{\langle 0 \rangle}^{(3)} \otimes s_{\langle 0 \rangle}^{(3)}] \end{aligned}$$

By (20),

$$m_A[r_{\langle 0 \rangle}^{(2)} \# r_{\langle -1 \rangle}^{(3)} \otimes s^{(2)} \# s_{\langle -1 \rangle}^{(3)}] = m[(r_{\langle 0 \rangle}^{(2)})^{(1)} \otimes (r_{\langle 0 \rangle}^{(2)})^{(2)} r_{\langle -1 \rangle}^{(3)} s^{(2)}] \# \xi((r_{\langle 0 \rangle}^{(2)})^{(2)} \otimes r_{\langle -2 \rangle}^{(3)} s^{(3)}) r_{\langle -1 \rangle}^{(3)} s_{\langle -1 \rangle}^{(4)},$$

and so $m^{v_A}(r \# 1 \otimes s \# 1)$ equals:

$$\begin{aligned} & v(r^{(1)} \otimes r_{\langle -1 \rangle}^{(2)} r_{\langle -2 \rangle}^{(3)} x^{(1)}) m(r_{\langle 0 \rangle}^{(2)} \otimes r_{\langle -1 \rangle}^{(3)} x^{(2)}) \# \xi(r_{\langle 0 \rangle}^{(3)} \otimes x^{(3)}) r_{\langle -1 \rangle}^{(4)} s_{\langle -1 \rangle}^{(4)} v^{-1}(r_{\langle 0 \rangle}^{(4)} \otimes s_{\langle 0 \rangle}^{(4)}) \\ & \quad \text{where } x := r_{\langle -2 \rangle}^{(4)} s^{(1)} \\ &= v(r^{(1)} \otimes r_{\langle -1 \rangle}^{(2)} (r_{\langle -1 \rangle}^{(3)} x^{(1)})^{(1)}) m(r_{\langle 0 \rangle}^{(2)} \otimes (r_{\langle -1 \rangle}^{(3)} x^{(1)})^{(2)}) \# \xi(r_{\langle 0 \rangle}^{(3)} \otimes x^{(2)}) r_{\langle -1 \rangle}^{(4)} s_{\langle -1 \rangle}^{(3)} v^{-1}[r_{\langle 0 \rangle}^{(4)} \otimes s_{\langle 0 \rangle}^{(3)}] \\ &= (v * m)(r^{(1)} \otimes r_{\langle -1 \rangle}^{(2)} r_{\langle -3 \rangle}^{(3)} s^{(1)}) \# \xi(r_{\langle 0 \rangle}^{(2)} \otimes r_{\langle -2 \rangle}^{(3)} s^{(2)}) r_{\langle -1 \rangle}^{(3)} s_{\langle -1 \rangle}^{(3)} v^{-1}(r_{\langle 0 \rangle}^{(3)} \otimes s_{\langle 0 \rangle}^{(3)}) \\ &= (v * m)(r^{(1)} \otimes r_{\langle -1 \rangle}^{(2)} r_{\langle -2 \rangle}^{(3)} s^{(1)}) \# \xi(r_{\langle 0 \rangle}^{(2)} \otimes r_{\langle -1 \rangle}^{(3)} s^{(2)}) \Psi(v^{-1})(r_{\langle 0 \rangle}^{(3)} \otimes s^{(3)}) \\ &= (v * m)(r^{(1)} \otimes r_{\langle -1 \rangle}^{(2)} s^{(1)}) \# \xi((r_{\langle 0 \rangle}^{(2)})^{(1)} \otimes (r_{\langle 0 \rangle}^{(2)})^{(2)} s_{\langle -1 \rangle}^{(2)}) \Psi(v^{-1})((r_{\langle 0 \rangle}^{(2)})^{(2)} \otimes s^{(3)}) \\ &= (v * m)(r^{(1)} \otimes r_{\langle -1 \rangle}^{(2)} s^{(1)}) \# [\xi * \Psi(v^{-1})](r_{\langle 0 \rangle}^{(2)} \otimes s^{(2)}) \\ &= (v * m)(r^{(1)} \otimes r_{\langle -1 \rangle}^{(2)} s^{(1)}) \# v^{-1}(r_{\langle 0 \rangle}^{(2)} \otimes s^{(2)}) \\ &= (v * m * v^{-1})(r \otimes s) \# 1 = m^v(r \otimes s) \# 1. \end{aligned}$$

Since Δ_A is H -bilinear, and both v_A and m_A are H -bilinear H -balanced, m^{v_A} is also H -bilinear H -balanced so that:

$$m^{v_A}(r \# h \otimes s \# l) = m^{v_A}(r \# 1 \otimes h_{(1)} s \# 1) h_{(2)} l = m^v(r \otimes h_{(1)} s) \# h_{(2)} l = m_{R^v \# H}(r \# h \otimes s \# l),$$

and $A^{vA} = R^v \# H$.

It remains to check that the reassociators are the same. Set $\alpha := \partial_R^2(v)$. Since A is a dual quasi-Hopf algebra with trivial reassociator ε , by (32), the reassociator γ for A^{vA} is:

$$\begin{aligned} \gamma &:= (\varepsilon_A \otimes v_A) * v_A (A \otimes m_A) * v_A^{-1} (m_A \otimes A) * (v_A^{-1} \otimes \varepsilon_A) \\ &= \partial_A^2(v_A) = \partial_A^2 \mathcal{U}_{H,R}^2(v) = \mathcal{U}_{H,R}^3 \partial_R^2(v) = \mathcal{U}_{H,R}^3(\alpha) \end{aligned}$$

so that

$$\begin{aligned} \gamma(r \# h \otimes s \# l \otimes t \# k) &= \mathcal{U}_{H,R}^3(\alpha)(r \# h \otimes s \# l \otimes t \# k) \\ &= \alpha(r \otimes h_{(1)} s \otimes h_{(2)} l t) \varepsilon_H(k) = \alpha_{R^v \# H}(r \# h \otimes s \# l \otimes t \# k). \end{aligned}$$

□

6. THE MAIN THEOREM

Recall from Section 2.4, that if (A, H, π, σ) is a splitting datum, then there is associated to this datum a pre-bialgebra with cocycle (R, ξ) . Furthermore, given a pre-bialgebra with cocycle (R, ξ) , then one can construct a splitting datum $(R \#_\xi H, H, \pi, \sigma)$ and $R \#_\xi H \cong A$. We can now prove the main result of this paper. Recall our assumption that H has an ad-invariant integral; for example H could be a group algebra.

THEOREM 6.1. *Let (A, H, π, σ) be a splitting datum with associated pre-bialgebra with cocycle (R, ξ) so that $A \cong B := R \#_\xi H$. Suppose $\sigma(H) = \text{Corad}(A)$. Then A is quasi-isomorphic to the bosonization of a connected dual quasi-bialgebra in ${}^H_H\mathcal{YD}$ by H .*

Proof. Let $(R, m, u, \Delta, \varepsilon, \xi)$ be the pre-bialgebra with cocycle in ${}^H_H\mathcal{YD}$ associated to (A, H, π, σ) . Since $\sigma(H) = \text{Corad}(A)$, by [Mo, Corollary 5.3.5], $\text{Corad}(R) \subseteq \tau(\text{Corad}(A)) \subseteq \tau(\sigma(H)) \subseteq K$ where $\tau(a) = a_{(1)} \sigma S_H \pi(a_{(2)})$ as described in Section 2.4. Thus $\text{Corad}(R) = K$ whence R is connected. Since H has an ad-invariant integral, by Theorem 4.3 we have the datum $(R, m^v, u, \Delta, \varepsilon, v)$ and by Proposition 5.5, $(R, m^v, u, \Delta, \varepsilon, \alpha)$ is a dual quasi-bialgebra in the braided monoidal category ${}^H_H\mathcal{YD}$. Set $B := R \#_\xi H$. By Proposition 5.7, there exists a gauge transformation $v_B : B \otimes B \rightarrow K$ such that

$$B^{v_B} = R^v \# H.$$

where the latter is the bosonization of the dual quasi-bialgebra $(R, m^v, u, \Delta, \varepsilon, \alpha)$ in ${}^H_H\mathcal{YD}$ by H . In conclusion A is quasi-isomorphic to the bosonization of the connected dual quasi-bialgebra $(R, m^v, u, \Delta, \varepsilon, \alpha)$ in ${}^H_H\mathcal{YD}$ by H . □

We can now give the proof of the main theorem.

Proof of Theorem I. By [AMSte, Theorem 2.35], A fits into a splitting datum (A, H, π, σ) where $\sigma : H \rightarrow A$ is the canonical inclusion. As mentioned in Section 2, since H is semisimple and cosemisimple, it has an ad-invariant integral so that Theorem 6.1 applies. □

REMARK 6.2. Let A be a bialgebra whose coradical H is a subbialgebra of A with antipode. Akira Masuoka pointed out, see [Mas], that, by Takeuchi's lemma [Mo, Lemma 5.2.10], A is necessarily a Hopf algebra. Thus A is a Hopf algebra with the dual Chevalley property.

Acknowledgement

We would like to thank the referee for an extremely helpful report, and especially for pointing out that our main result could be explained more conceptually by the use of Diagram (1).

REFERENCES

- [ABM] A. Ardizzoni, M. Beattie and C. Menini, *Cocycle Deformations for Hopf Algebras with a Coalgebra Projection*, J. Algebra, Vol. **324**(4) (2010), 673–705.
- [ABM1] A. Ardizzoni, M. Beattie and C. Menini, *Cocycle Deformations for liftings of quantum linear spaces*, Comm. Algebra, to appear. (arXiv:1011.0648)
- [AM] A. Ardizzoni and C. Menini, *Small Bialgebras with a Projection: Applications*, Comm. Algebra, Vol. **37**(8) (2009), 2742–2784.

- [AMSte] A. Ardizzoni, C. Menini and D. Stefan, *A Monoidal Approach to Splitting Morphisms of Bialgebras*, Trans. Amer. Math. Soc., **359** (2007), 991–1044.
- [AMStu] A. Ardizzoni, C. Menini and F. Stumbo, *Small Bialgebras with Projection*, J. Algebra, **314**(2) (2007), 613–663.
- [Ar] A. Ardizzoni, *Separable functors and formal smoothness*, J. K-Theory **1** (2008), no. 3, 535–582.
- [BC] J. Bichon, G. Carnovale, *Lazy cohomology: an analogue of the Schur multiplier for arbitrary Hopf algebras*, J. Pure & Appl. Algebra **204** (2006), 627–665.
- [BN] D. Bulacu, E. Nauwelaerts, *Radford’s biproduct for quasi-Hopf algebras and bosonization*, J. Pure Appl. Algebra **174** (2002), no. 1, 1–42.
- [Doi] Y. Doi, *Braided bialgebras and quadratic bialgebras*, Comm. Algebra **21** (1993), no. 5, 1731–1749.
- [Ka] C. Kassel, *Quantum groups*, Graduate Text in Mathematics **155**, Springer, 1995.
- [La] R. G. Larson, *Characters of Hopf algebras*, J. Algebra **17** (1971), 352–368.
- [Maj] S. Majid, *Foundations of quantum group theory*, Cambridge University Press, 1995.
- [Mas] A. Masuoka, private communication.
- [Mo] S. Montgomery, *Hopf Algebras and their actions on rings*, CMBS Regional Conference Series in Mathematics **82**, 1993.
- [Scha] P. Schauenburg, *The structure of Hopf algebras with a weak projection*, Algebr. Represent. Theory **3** (2000), 187–211.
- [SvO] D. Stefan and F. van Oystaeyen, *The Wedderburn-Malcev theorem for comodule algebras*, Comm. Algebra **27** (1999), 3569–3581.
- [Sw] M. Sweedler, *Hopf Algebras*, Benjamin, New York, 1969.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF FERRARA, VIA MACHIAVELLI 35, FERRARA I-44121, ITALY

E-mail address: `alessandro.ardizzoni@unife.it`

URL: `http://www.unife.it/utenti/alessandro.ardizzoni`

DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE, MOUNT ALLISON UNIVERSITY, SACKVILLE, NB, CANADA, E4L 1E6

E-mail address: `mbeattie@mta.ca`

URL: `http://www.mta.ca/~mbeattie/`

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF FERRARA, VIA MACHIAVELLI 35, FERRARA I-44121, ITALY

E-mail address: `men@dns.unife.it`

URL: `http://www.unife.it/utenti/claudia.menini`